Endoscopic character identities for depth-zero supercuspidal L-packets

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In this paper we prove the conjectural endoscopic character identities for the local Langlands correspondence constructed in [DR09]. The local Langlands correspondence, which is known in the real case and partially constructed in the p-adic case, assigns to each Langlands parameter for a reductive group G over a local field F a finite set of admissible irreducible representations of $G(F)$, called an L-packet. When such a parameter factors through an endoscopic group H, the broad principle of Langlands functoriality asserts that the packet on H should "transfer" to the packet on G. The endoscopic character identities are an instance of this principle – they state that the "stable" character of the packet on H is identified via endoscopic induction with an "unstable" character of the packet on G.

To be more precise, let F be a p-adic field with Weil-group W_F and let G be a connected reductive group over F. For the purposes of this introduction, we assume that G is unramified, although in the body of this paper the more general case of a pure inner form of an unramified group is handled. Let ${}^LG$ be an L-group for G, that is ${}^LG = \widehat{G} \rtimes W_F$, where $\widehat{G}$ is the complex Langlands dual of G and W_F acts on $\widehat{G}$ via its action on the based root datum of $\widehat{G}$ which is dual to that of G. The Langlands parameters considered in this paper are continuous sections

$$W_F \to {}^LG$$

of the natural projection ${}^LG \to W_F$ and subject to certain conditions, called TRSELP in [DR09], which will be reviewed in detail later on. To such a parameter DeBacker and Reeder construct in loc.cit. an L-packet $\Pi_G(\varphi)$ of representations of $G(F)$ and a bijection

$$\mathrm{Irr}(C_\varphi, 1) \to \Pi_G(\varphi), \qquad \rho \mapsto \pi_\rho$$

where C_φ is the component group of the centralizer in $\widehat{G}$ of φ and $\mathrm{Irr}(C_\varphi, 1)$ are those representations of the finite group C_φ which are trivial on elements of C_φ coming from the center of $\widehat{G}$. This bijection maps the trivial representation of C_φ to a generic representation of $G(F)$.

Let $(H, s, \widehat{\eta})$ be an unramified endoscopic triple for G. Recall that H is an unramified reductive group over F, s is a Galois-fixed element of the center of $\widehat{H}$, and $\widehat{\eta}$ is an inclusion $\widehat{H} \to \widehat{G}$ which identifies $\widehat{H}$ with $(\widehat{G}_{\widehat{\eta}(s)})^\circ$. It was shown by Hales that $\widehat{\eta}$ extends to an embedding ${}^L\eta : {}^LH \to {}^LG$. Thus for any parameter φ^H for H we may consider the parameter $\varphi = {}^L\eta \circ \varphi^H$, i.e. we have

$$\begin{tikzcd}
{}^LH \arrow[r, "{}^L\eta"] & {}^LG \\
W_F \arrow[u, "\varphi^H"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

If both parameters are of the type considered here, then we have the L-packets $\Pi_G(\varphi)$ and $\Pi_H(\varphi^H)$. Associated to these, we have the stable character

$$\mathcal{S}\Theta_{\varphi^H} := \sum_{\rho \in \mathrm{Irr}(C_{\varphi^H},1)} [\dim \rho] \chi_{\pi_\rho}$$

of $\Pi_H(\varphi^H)$, which is a stable function on $H(F)$ (this is one of the main results of [DR09]), as well as the s-unstable character

$$\Theta^s_{\varphi,1} := \sum_{\rho \in \mathrm{Irr}(C_\varphi,1)} [\mathrm{tr}\, \rho(s)] \chi_{\pi_\rho}$$

of $\Pi_G(\varphi)$, which is an invariant function on $G(F)$.

Recall that the representation π_1 of $G(F)$ is generic. Thus there is a Borel subgroup $B = TU$ of G defined over F and a generic character $\psi : U(F) \to \mathbb{C}^\times$ which occurs in the restriction of π_1 to $U(F)$. Associated to the character ψ there is a unique normalization Δ_ψ of the transfer factor for G and H, called the Whittaker normalization. The endoscopic lift of the stable function $\mathcal{S}\Theta_{\varphi^H}$ is given by

$$\mathrm{Lift}^G_H \mathcal{S}\Theta_{\varphi^H}(\gamma) := \sum_{\gamma^H} \Delta_\psi(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H)$$

where $\gamma \in G(F)$ is any strongly regular semi-simple element and γ^H runs through the set of stable classes of G-strongly regular semi-simple elements in $H(F)$.

The main result of this paper asserts that

$$\Theta^s_{\varphi,1} = \mathrm{Lift}^G_H \mathcal{S}\Theta_{\varphi^H}$$

As a corollary of the main result in the case where G is a pure inner form of an unramified group G^* and $H = G^*$ we obtain a proof (for the L-packets considered) of the conjecture of Kottwitz [Kot83] about sign changes in stable characters on inner forms.

We now describe the contents of the paper. After fixing some basic notation in Section 1, we discuss pure inner twists and the associated notions of conjugacy and stable conjugacy. We have allowed trivial inner twists in the discussion so as to accommodate the natural construction of the L-packets in [DR09] and not just their normalized form. With these notions in place we implement an observation of Kottwitz which allows one to define compatible normalizations of the absolute transfer factors for all pure inner twists. In Section 3 we briefly review the construction of the local Langlands correspondence in [DR09], and after gathering the necessary notation we state the main result of this paper. The remaining sections are devoted to its proof, which is similar in spirit to the proof of the stability result in loc. cit.. In Section 4 we study three signs which are defined for a pair (G, H) of a group G and an endoscopic group H and play an important role in the theory of endoscopy – one of them is defined in terms of the split ranks of these groups and goes back to [Kot83], the other one occurs in Waldspurger's work [Wal95] on the endoscopic transfer for p-adic Lie algebras, and the third is a certain local ϵ-factor used in the Whittaker normalization of the transfer factors [KS99]. We show that when both G and H are unramified, these three signs coincide. This supplements the results of [DR09, §12] to assert in particular that the Waldspurger-sign and the

relative-ranks-sign coincide whenever G is a pure inner form of an unramified group and H is an unramified endoscopic group. Because this section may be of independent interest we have minimized the notation that it borrows from previous sections. Section 5 deals with establishing a reduction formula for the unstable character of an L-packet with respect to the topological Jordan decomposition. For that we first need explicit formulas for some basic constructions in endoscopy, which are established in two preparatory subsections. Among other things we show that the isomorphism $H^1(F,G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$ constructed in [DR09] via Bruhat-Tits theory coincides with the one constructed in [Kot86] using Tate-Nakayama duality. With these preliminaries in place we derive the reduction formula for the unstable character using the results of [DR09, §9,§10]. The ingredients from the previous sections are combined in Section 6 to establish the proof of the main result. After reducing to the case of compact elements the reduction formula from Section 5 is combined with the work of Langlands and Shelstad [LS90] and Hales [Hal93] on endoscopic descent. The topologically unipotent part of the resulting expression is then transferred to the Lie algebra, where we invoke the deep results of Waldspurger on endoscopic transfer for p-adic Lie-algebras together with the fundamental lemma, which has been recently proved by the combined effort of many people.

We would like to bring to the attention of the reader some related work on this problem. In [KV1], Kazhdan and Varshavsky construct an endoscopic decomposition for the L-packets considered here. In particular, they consider the s-unstable characters of these packets and show that they belong to a space of functions which contains the image of endoscopic induction. The existence of such a decomposition is a necessary condition for the validity of the character identities considered here and also gave us yet more reason to hope that indeed these identities should be true. In [KV2] the aforementioned authors prove a formula for the geometric endoscopic transfer of Deligne-Lusztig functions, in particular answering a conjecture of Kottwitz. After the current paper was written, the author was informed in a private conversation with Kazhdan that the results in [KV2] could likely be used to derive character identities similar to the ones proved here, at least on the set of elliptic elements, and possibly in general.

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CONTENTS

1 Notation

Let F be a p-adic field (i.e. a finite extension of $\mathbb{Q}_p$) with ring of integers O_F, uniformizer π_F, and residue field $k_F = O_F/\pi_F O_F$ with cardinality q_F. We use analogous notation for any other discretely valued field, in particular for the maximal unramified extension F^u of F in a fixed algebraic closure $\overline{F}$. Since we will consider only extensions of F which lie in F^u, π_F will be a uniformizer in each of them and so we will drop the index F and simply call it π. For any such finite extension E, $v_E : E^\times \to \mathbb{Z}$ will be the discrete valuation normalized so that $v_E(\pi) = 1$, and $|x|_E$ will be the norm given by $q_E^{-v_E(x)}$. Thus v_E extends v_F and so we may again drop the index F. On the other hand, for $x \in F^\times$ we have $|x|_E = |x|_F^{[E:F]}$; if dx is any additive Haar measure on E then $d(ax) = |a|_E dx$. The absolute Galois group of F will be denoted by Γ, its Weil group by W_F and inertia group by I_F. We choose an element $\mathrm{Fi} \in \Gamma$ whose inverse induces on $\overline{k_F}$ the map $x \mapsto x^{q_F}$.

For a reductive group G defined over F, we will denote its Lie algebra by the Fraktur letter $\mathfrak{g}$. Our convention will be that $a \in G$ resp. $a \in \mathfrak{g}$ will mean that a is an $\overline{F}$-point of the corresponding space, while a maximal torus $T \subset G$ will be tacitly assumed to be defined over F. The action of Fi on both $G(F^u)$ and $\mathfrak{g}(F^u)$ will be denoted by Fi_G. For a semi-simple $a \in G$, we will write $\mathrm{Cent}(a, G) = G^a$ for the centralizer of a in G and G_a for its connected component. If $T \subset G$ is a maximal torus then the roots resp. coroots of T in G will be denoted by $R(T, G)$ resp. $R^\vee(T, G)$. The center of G will be Z_G, or simply Z if G is understood, and the maximal split torus in Z_G will be A_G. The sets of strongly-regular semi-simple elements of G resp. $\mathfrak{g}$ will be denoted by G_{sr} resp. $\mathfrak{g}_{\mathrm{sr}}$. The set of compact elements in $G(F)$ will be denoted by $G(F)_0$ (note that we are using the wording of [DR09] here; in [Hal93] these elements are called strongly-compact). For any $g \in G$ the map $G \to G, x \mapsto gxg^{-1}$ as well as its tangent map $\mathfrak{g} \to \mathfrak{g}$ will be called $\mathrm{Ad}(g)$. Abusing words, will will refer to the orbits of $\mathrm{Ad}(G)$ in $\mathfrak{g}$ as conjugacy classes, and then notions such as stable classes and rational classes will have their obvious meaning.

To maintain notational similarity with [DR09], we will sometimes use the following conventions. If $\psi : G \to G'$ is an inner twist, then we may identify $G(\overline{F})$ and $G'(\overline{F})$ via ψ and suppress ψ from the notation, thereby treating $\gamma \in G(\overline{F})$ and $\psi(\gamma) \in G'(\overline{F})$ as the same element. If $u \in Z^1(\Gamma, G)$ is a cocycle, then we will use the same letter u also for the value of that cocycle at Fi.

If $(H, s, \widehat{\eta})$ is an endoscopic triple for a reductive group G/F, we will often attach a superscript H to objects related to H, such as maximal tori, Borels, or elements of $H(\overline{F})$. If ${}^L\eta : {}^LH \to {}^LG$ is an L-embedding extending $\widehat{\eta}$, then we will call $(H, s, {}^L\eta)$ an extended triple for G. The set of G-strongly regular semi-simple elements of H resp. $\mathfrak{h}$ will be denoted by $H_{G-\mathrm{sr}}$ resp. $\mathfrak{h}_{G-\mathrm{sr}}$. Let $t^H \in H(F)$ and $t \in G(F)$ be semi-simple elements. We will call t an image of t^H if there exist maximal tori $T^H \subset H$ and $T \subset G$ and an admissible isomorphism $T^H \to T$ defined over F and mapping t^H to t. This definition is the same as in [LS90], but our wording is opposite – in [LS90] the element t^H is called an image of t. If t is an image of t^H we will also call (t^H, t) a pair of related elements. For such a pair, we consider the set of $\varphi : T^H \to T$, where T^H is a maximal torus in H containing t^H, T is a maximal torus of G containing t, and φ is an admissible isomorphism defined over F and mapping t^H to t. On this set we define an equivalence relation, by saying that two such isomorphisms φ and φ' are (G_t, H_{t^H})-equivalent if there exist $g \in G_t(\overline{F})$ and $h \in H_{t^H}(\overline{F})$ s.t.

$\varphi' = \mathrm{Ad}(g)\varphi\mathrm{Ad}(h)$. If φ is an element of this set, and H_{t^H} is quasi-split, then H_{t^H} can be identified with an endoscopic group of G_t in such a way that φ becomes an admissible isomorphism with respect to (G_t, H_{t^H}). Then we can talk about images, admissible isomorphisms, etc. with respect to the group G_t and its endoscopic group H_{t^H}. When we do so, we will use the prefix (G_t, H_{t^H}, φ).

If γ, γ' are two strongly G-regular semi-simple elements, each of which belongs to either $G(F)$ or $H(F)$, and T, T' are their centralizers, then there exists at most one admissible isomorphism $T \to T'$ which maps γ to γ'. We will call this isomorphism $\varphi_{\gamma,\gamma'}$. If it exists, then so does $\varphi_{\gamma',\gamma}$ and $\varphi_{\gamma',\gamma} = \varphi_{\gamma,\gamma'}^{-1}$. Moreover, if $\gamma, \gamma', \gamma''$ are three elements as above and $\varphi_{\gamma,\gamma'}$ and $\varphi_{\gamma',\gamma''}$ exist, then so does $\varphi_{\gamma,\gamma''}$ and

$$\varphi_{\gamma,\gamma''} = \varphi_{\gamma',\gamma''} \circ \varphi_{\gamma,\gamma'}$$

The same can also be done with regular semi-simple elements of the Lie algebras of G and H and we will use the same notation for that case.

2 Pure inner twists

Let A, B be reductive groups over F. A pure inner twist

$$(\psi, z) : A \to B$$

consists of an isomorphism of $\overline{F}$-groups $\psi : A \times \overline{F} \to B \times \overline{F}$ and an element $z \in Z^1(\Gamma, A)$ s.t.

$$\forall \sigma \in \Gamma : \psi^{-1}\sigma(\psi) = \mathrm{Ad}(z_\sigma)$$

We will from now on abbreviate "pure inner twist" to simply "twist", since these will be the only twists of reductive groups that will concern us here.

The twist (ψ, z) is called trivial the image of z in $H^1(\Gamma, A)$ is trivial. In that case there exists $a \in A(\overline{F})$ s.t.

$$\psi \circ \mathrm{Ad}(a) : A \to B$$

is an isomorphism over F. Clearly the element a is unique up to right multiplication by $A(F)$. We will call the twist (ψ, z) strongly trivial if $z = 1$. In that case of course ψ is already defined over F. An example of a trivial twist is given by $(\mathrm{Ad}(g), g^{-1}\sigma(g)) : A \to A$ for any $g \in A(\overline{F})$. This twist is strongly trivial if and only if $g \in A(F)$.

Starting from $(\psi, z) : A \to B$ we can form the inverse twist $(\psi, z)^{-1} : B \to A$, which is given by $(\psi^{-1}, \psi(z_\sigma^{-1}))$.

If $(\psi, z) : A \to B$ and $(\varphi, u) : B \to C$ are twists, then we can form their composition

$$(\varphi, u) \circ (\psi, z) : A \to C$$

which is given by $(\varphi \circ \psi, \psi^{-1}(u)z)$. One immediately checks

$$\begin{aligned}
(\psi, z) \circ (\psi, z)^{-1} &= (\mathrm{id}_B, 1) \\
(\psi, z)^{-1} \circ (\psi, z) &= (\mathrm{id}_A, 1) \\
[(\varphi, u) \circ (\psi, z)]^{-1} &= (\psi, z)^{-1} \circ (\varphi, u)^{-1} \\
(\chi, v) \circ [(\varphi, u) \circ (\psi, z)] &= [(\chi, v) \circ (\varphi, u)] \circ (\psi, z)
\end{aligned}$$

In particular, reductive groups and pure inner twists form a groupoid.

Let $(\psi, z), (\psi', z') : A \to B$ be two twists. They are called equivalent if $(\psi', z') \circ (\psi, z)^{-1}$ equals $(\mathrm{Ad}(g), g^{-1}\sigma(g))$ for some $g \in B(\overline{F})$. One immediately checks the equality

$$(\psi, z)^{-1} \circ (\mathrm{Ad}(g), g^{-1}\sigma(g)) \circ (\psi, z) = (\mathrm{Ad}(h), h^{-1}\sigma(h)), \quad h = \psi^{-1}(g)$$

from which it follows that this defines an equivalence relation on all inner twists which is invariant under composition and taking inverses.

2.1 Conjugacy along pure inner twists

Now consider a twist $(\psi, z) : A \to B$ and two elements $a \in A(F), b \in B(F)$. We call a, b conjugate (with respect to (ψ, z)) if there exists a twist (ψ', z') equivalent to (ψ, z) which maps a to b and is strongly trivial. We call a, b stably conjugate (with respect to (ψ, z)) if there exists a twist (ψ', z') equivalent to (ψ, z) which maps a to b and descends to a twist $A_a \to B_b$. The latter condition simply means that z' takes values in B_b (a-priori it only takes values in $\mathrm{Cent}(b, B)$).

The following is immediately clear

Fact 2.1.1.

1. *Applied to the twist $(\mathrm{id}, 1) : A \to A$ the notions defined above coincide with the usual ones for the group A*

2. *If $a \in A(F), b \in B(F)$ are conjugate with respect to $(\psi, z) : A \to B$, then they are also stably conjugate and moreover (ψ, z) is a trivial twist*

3. *If $a \in A(F), b \in B(F)$ are conjugate (resp. stably conjugate) with respect to $(\psi, z) : A \to B$, then so are they with respect to any twist equivalent to (ψ, z).*

4. *If $a \in A(F)$ and $b \in B(F)$ are conjugate (resp. stably-conjugate) with respect to $(\psi, z) : A \to B$, then so are they with respect to $(\psi, z)^{-1} : B \to A$*

5. *If $(\psi, z) : A \to B$ and $(\varphi, u) : B \to C$ are two twists and $a \in A(F), b \in B(F), c \in C(F)$ are s.t. a, b and b, c are conjugate (resp. stably-conjugate), then so are a, c.*

Let $a \in A(F)$ and $b \in B(F)$ be stably conjugate assume that $\mathrm{Cent}(a, A)$ is connected. Choose a twist $(\varphi, u) : A \to B$ which is equivalent to (ψ, z) and sends a to b, and write $\mathrm{inv}(a, b)$ for the image of u in $H^1(\Gamma, A_a)$.

Fact 2.1.2.

1. *The element $\mathrm{inv}(a, b)$ is independent of the choice of the twist (φ, u).*

2. *Applied to the twist $(\mathrm{id}, 1) : A \to A$, inv coincides with the usual definition for the group A.*

3. *The image of $\mathrm{inv}(a, b)$ in $H^1(\Gamma, A)$ equals z.*

Proof: This is obvious. $\square$

Fact 2.1.3. *Let $a \in A(F)$, $b \in B(F)$ and $c \in C(F)$ be s.t. the inner twists*

$$A \xrightarrow{(\varphi,u)} B \xrightarrow{(\psi,z)} C$$

send a to b to c. Assume that $\mathrm{Cent}(a, A)$ *is connected. Then*

$$\mathrm{inv}(a, c) = \varphi^{-1}(\mathrm{inv}(b, c))\mathrm{inv}(a, b)$$

Proof: This follows at once from the composition formula for twists. □

Now let A be quasi-split. We consider a set I of triples (A^z, ψ_z, z) s.t. $(\psi_z, z) : A \to A^z$ is a twist. Put

$$A^I = \bigsqcup_{(A^z,\psi_z,z)} A^z$$

This is a variety over F (it will not be of finite type if I is infinite). For $a \in A^z$ and $b \in A^{z'}$ we obtain notions of conjugacy and stable conjugacy, namely those relative to the twist $(\psi_{z'}, z') \circ (\psi_z, z)^{-1}$. Thus we can talk about conjugacy classes and stable conjugacy classes of elements of $A^I(F)$. For the sake of abbreviation, we will call a twist $(\varphi, u) : A^z \to A^{z'}$ allowable if it is equivalent to $(\psi_{z'}, z') \circ (\psi_z, z)^{-1}$. Note that the set of allowable twists is invariant under composing and taking inverses.

Fact 2.1.4. *Every stable conjugacy class of $A^I(F)$ meets $A(F)$.*

Proof: This is a consequence of a well known theorem of Steinberg, which implies that any maximal torus of a reductive group transfers to its quasi-split inner form. □

Lemma 2.1.5. *Let $\bar{I}$ be the image of I in $H^1(\Gamma, A)$ under the map $(A^z, \psi_z, z) \mapsto [z]$. Then for each $a \in A(F)$ whose centralizer is connected, the map $b \mapsto \mathrm{inv}(a, b)$ is a bijection from the set of conjugacy classes inside the stable class of a in $A^I(F)$ to the preimage of $\bar{I}$ under $H^1(\Gamma, A_a) \to H^1(\Gamma, A)$.*

Remark: One can prove a similar lemma for $a \in A^{z'}(F)$ and any z', but the statement is more awkward and we will not need it.

Proof: Let $b \in A^z(F)$ and $b' \in A^{z'}(F)$ be conjugate elements belonging to the stable class of a. Thus there exists an allowable strongly trivial twist $(\chi, 1) : A^z \to A^{z'}$ mapping b to b'. Let $(\varphi, u) : A \to A^z$ be an allowable twist mapping a to b, thus $\mathrm{inv}(a, b) = [u]$. Then $(\chi, 1) \circ (\varphi, u)$ is an allowable twist $A \to A^{z'}$, mapping a to b', so $\mathrm{inv}(a, b')$ equals the class of the cocycle of $[(\chi, 1) \circ (\varphi, u)]$, which is also $[u]$. This shows that $\mathrm{inv}(a, b) = \mathrm{inv}(a, b')$ and we see that the map $b \mapsto \mathrm{inv}(a, b)$ is a well-defined map on the set of conjugacy classes inside the stable class of a. By above facts it lands in the preimage of $\bar{I}$. We will show that it is injective. To that end, let $b \in A^z(F)$ and $b' \in A^{z'}(F)$ be s.t. $\mathrm{inv}(a, b) = \mathrm{inv}(a, b')$. Let $(\varphi, u) : A \to A^z$ and $(\varphi', u') : A \to A^{z'}$ be allowable twists sending a to b reps. b'. By assumption there exists $i \in A_a$ s.t. $u = i^{-1}u'\sigma(i)$. But then $(\varphi', u') \circ (\mathrm{Ad}(i), i^{-1}\sigma(i))$ is again an allowable twist $A \to A^{z'}$ sending a to b', and so replacing (φ', u') by it we achieve $u = u'$. But now it is clear that $(\varphi', u') \circ (\varphi, u)^{-1}$ is an allowable strongly trivial twist $A^z \to A^{z'}$ sending b to b', thus showing that b and b' are conjugate. Finally we show that the map $b \mapsto \mathrm{inv}(a, b)$ is surjective. Thus let $[u] \in H^1(\Gamma, A_a)$ be an element mapping to $[z] \in H^1(\Gamma, A)$, where $(A^z, \psi_z, z) \in I$. Then there exists $g \in A$ s.t. $u = g^{-1}z\sigma(g)$. Put $b = \psi_z(\mathrm{Ad}(g)a)$. One computes immediately that $b \in A^z(F)$. By construction $(\psi_z, z) \circ (\mathrm{Ad}(g), g^{-1}\sigma(g))$ maps a to b, which shows that a and b are stably conjugate and that $\mathrm{inv}(a, b) = g^{-1}z\sigma(g)$. □

2.2 Transfer factors for pure inner twists

Let G be a quasi-split F-group, $(\psi, z) : G \to G'$ a twist, and $(H, s, {}^L\eta)$ an extended triple for G. Then $(H, s, {}^L\eta)$ is also an extended triple for G'. This data gives canonical relative geometric transfer factors $\Delta_H^G(\gamma^H, \gamma, \bar\gamma^H, \bar\gamma)$ for (G, H) and $\Delta_H^{G'}(\gamma^H, \gamma', \bar\gamma^H, \bar\gamma')$ for (G', H) (see [LS87]). Let $\Delta_H^G(\gamma^H, \gamma)$ be an arbitrary normalization for the absolute transfer factor for (G, H). For any pair $\gamma^H \in H(F)$ and $\gamma' \in G'(F)$ of strongly G-regular related elements we choose an element $\gamma \in G(F)$ stably conjugate to γ' (which exists by Fact 2.1.4) and define

$$\Delta_H^{G'}(\gamma^H, \gamma') = \Delta_H^G(\gamma^H, \gamma) \cdot \langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}$$

where

$$\langle\rangle : H^1(\Gamma, T) \times \pi_0(\widehat{T}^\Gamma) \to \mathbb{C}^\times$$

is the Tate-Nakayama pairing, and $T = \mathrm{Cent}(\gamma, G)$.

Lemma 2.2.1. $\Delta_H^{G'}(\cdot,\cdot)$ *is well defined and is an absolute transfer factor for* (G', H)

Proof: We need to show that $\Delta_H^{G'}(\gamma^H, \gamma')$ is independent of the choice of γ. Thus let $\tilde\gamma \in G(F)$ be another element in the stable class of γ'. We know from [LS87]

$$\Delta_H^G(\gamma^H, \tilde\gamma) = \Delta_H^G(\gamma^H, \gamma)\langle \mathrm{inv}(\gamma, \tilde\gamma), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}$$

On the other hand if $(\varphi, u) : A \to A$ is an admissible twist mapping $\tilde\gamma$ to γ, then $\varphi_{\tilde\gamma,\gamma^H} = \varphi_{\gamma,\gamma^H} \circ \varphi$ and by functoriality of the Tate-Nakayama pairing we get

$$\langle \mathrm{inv}(\tilde\gamma, \gamma'), \widehat{\varphi}_{\tilde\gamma,\gamma^H}(s)\rangle^{-1} = \langle \varphi(\mathrm{inv}(\tilde\gamma, \gamma')), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}$$

Thus

$$\begin{aligned}
&\Delta_H^G(\gamma^H, \tilde\gamma)\langle \mathrm{inv}(\tilde\gamma, \gamma'), \widehat{\varphi}_{\tilde\gamma,\gamma^H}(s)\rangle^{-1} = \\
&\Delta_H^G(\gamma^H, \gamma)\langle \mathrm{inv}(\gamma, \tilde\gamma), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1} \, \langle \varphi(\mathrm{inv}(\tilde\gamma, \gamma')), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1} = \\
&\Delta_H^G(\gamma^H, \gamma)\langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}
\end{aligned}$$

by Fact 2.1.3.

This shows that $\Delta_H^{G'}(\gamma^H, \gamma')$ is independent of the choice of γ. To show that it is an absolute transfer factor for (G', H) we must prove for any two strongly G-regular related pairs (γ^H, γ') and $(\bar\gamma^H, \bar\gamma')$ in $H(F) \times G'(F)$ the equality

$$\frac{\Delta_H^{G'}(\gamma^H, \gamma')}{\Delta_H^{G'}(\bar\gamma^H, \bar\gamma')} = \Delta_H^{G'}(\gamma^H, \gamma', \bar\gamma^H, \bar\gamma')$$

which by construction of $\Delta_H^{G'}$ is equivalent to

$$\frac{\langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}}{\langle \mathrm{inv}(\bar\gamma, \bar\gamma'), \widehat{\varphi}_{\bar\gamma,\bar\gamma^H}(s)\rangle^{-1}} = \frac{\Delta_H^{G'}(\gamma^H, \gamma', \bar\gamma^H, \bar\gamma')}{\Delta_H^G(\gamma^H, \gamma, \bar\gamma^H, \bar\gamma)}$$

where $\gamma \in G(F)$ is any element in the stable class of γ', and $\bar\gamma \in G(F)$ is any element in the stable class of $\bar\gamma'$. Applying [LS87, Lemma 4.2.A] we need to show

$$\frac{\langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}}{\langle \mathrm{inv}(\bar\gamma, \bar\gamma'), \widehat{\varphi}_{\bar\gamma,\bar\gamma^H}(s)\rangle^{-1}} = \left\langle \mathrm{inv}\left(\frac{\gamma, \gamma'}{\bar\gamma, \bar\gamma'}\right), s_U \right\rangle$$

The right hand side of this equality is constructed in [LS87, §3.4]. Working through the construction, one sees that in our case the objects are as follows: Let T and $\overline{T}$ denote the centralizers of γ and $\bar{\gamma}$ in G, and let $T_{\rm sc}$ and $\overline{T}_{\rm sc}$ be their preimages in the simply connected cover $G_{\rm sc}$ of the derived group of G. Let $Z_{\rm sc}$ be the center of $G_{\rm sc}$. Then $U = T_{\rm sc} \times \overline{T}_{\rm sc}/\{(z^{-1}, z)|\ z \in Z_{\rm sc}\}$. We have the following dual diagrams

$$T \times \overline{T} \longleftarrow T_{\rm sc} \times \overline{T}_{\rm sc} \longrightarrow U$$

$$\widehat{T} \times \widehat{\overline{T}} \longrightarrow \widehat{T}_{\rm ad} \times \widehat{\overline{T}}_{\rm ad} \longleftarrow \widehat{U}$$

The elements $s_U \in \widehat{U}^\Gamma$ and $(\widehat{\varphi}_{\gamma^H,\gamma}(s), \widehat{\varphi}_{\bar{\gamma}^H,\bar{\gamma}}(s)) \in \widehat{T} \times \widehat{\overline{T}}$ map to the same element in $\widehat{T}_{\rm ad} \times \widehat{\overline{T}}_{\rm ad}$. There exists an element of $H^1(\Gamma, T_{\rm sc}) \times H^1(\Gamma, \overline{T}_{\rm sc})$ which maps to $\mathrm{inv}\left(\frac{\gamma,\gamma'}{\bar{\gamma},\bar{\gamma}'}\right) \in H^1(\Gamma, U)$ and to $(\mathrm{inv}(\gamma,\gamma')^{-1}, \mathrm{inv}(\bar{\gamma},\bar{\gamma}')) \in H^1(\Gamma, T) \times H^1(\Gamma, \overline{T})$. The equality now follows again from the functoriality of the Tate-Nakayama pairing. $\square$

Now let I be a set of pure inner twists for G and construct G^I as above. Taking the disjoint union over I of all functions $\Delta_H^{G^z}$ we obtain a function

$$\Delta_H^{G^I} : H_{G-\rm sr}(F) \times G^I_{\rm sr}(F) \longrightarrow \mathbb{C}^\times$$

Fact 2.2.2. *For all stably conjugate $\gamma, \gamma' \in G^I_{\rm sr}(F)$ and $\gamma^H \in H_{G-\rm sr}(F)$ we have*

$$\Delta_H^{G^I}(\gamma^H, \gamma') = \Delta_H^{G^I}(\gamma^H, \gamma) \cdot \langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma,\gamma^H}(s)\rangle^{-1}$$

Proof: Let $\gamma_0 \in G(F)$ be an element stably conjugate to γ (it exists by Fact 2.1.4). Then by construction of $\Delta_H^{G^I}$ we have

$$\Delta_H^{G^I}(\gamma^H, \gamma')\Delta_H^{G^I}(\gamma^H, \gamma)^{-1} = \langle \mathrm{inv}(\gamma_0, \gamma')\mathrm{inv}(\gamma_0, \gamma)^{-1}, \widehat{\varphi}_{\gamma_0,\gamma_H}(s)\rangle^{-1}$$

By Fact 2.1.3 the right hand side equals $\langle \varphi_{\gamma,\gamma_0}(\mathrm{inv}(\gamma,\gamma')), \widehat{\varphi}_{\gamma_0,\gamma_H}(s)\rangle^{-1}$ and the claim now follows from the functoriality of the Tate-Nakayama pairing. $\square$

Remark: We see in particular the the function $\gamma \mapsto \Delta_H^{G^I}(\gamma^H, \gamma)$ is constant on the conjugacy classes of $G^I(F)$.

3 Statement of the main result

We fix an unramified reductive group G over F, and a Borel pair (T_0, B_0) of G defined over F. Then Γ acts on $X^*(T_0)$ through a finite cyclic subgroup of $\mathrm{Aut}(X^*(T_0))$ generated by the image of Fi; we will denote by ϑ both this image as well as its dual in $\mathrm{Aut}(X_*(T_0))$. Let $(\widehat{G}, \widehat{B}_0, \widehat{T}_0)$ be the dual datum to (G, B_0, T_0). If $\Omega(T_0, G)$ and $\Omega(\widehat{T}_0, \widehat{G})$ denote the corresponding Weyl-groups, then there is a natural isomorphism between them given by duality. We choose an L-group ${}^LG$ for G s.t. the Γ-action on $\widehat{G}$ preserves the pair $(\widehat{B}_0, \widehat{T}_0)$.

We also fix an endoscopic triple $(H, s, \widehat{\eta})$ for G s.t. H is unramified. We choose again a Borel pair (T_0^H, B_0^H) defined over F, let $(\widehat{H}, \widehat{B}_0^H, \widehat{T}_0^H)$ be the dual datum

to (H, B_0^H, T_0^H) and ${}^LH$ an L-group for H s.t. the Γ-action on $\widehat{H}$ preserves $(\widehat{B}_0^H, \widehat{T}_0^H)$.

We choose a hyperspecial point o in the apartment of T_0 and obtain an O_F-structure on G and $\mathfrak{g}$. Then G_o, G_{o^+} resp. $\mathfrak{g}_o, \mathfrak{g}_{o^+}$ will be the parahoric and its pro-unipotent radical of $G(O_{F^u})$ resp. $\mathfrak{g}(O_{F^u})$ associated to o. We also choose a hyperspecial point, denoted again by o, in the apartment of T_0^H and obtain the same structures on H and $\mathfrak{h}$.

Up to equivalence the map $\widehat{\eta} : \widehat{H} \to \widehat{G}$ may be chosen so that $\widehat{\eta}^{-1}(\widehat{B}_0, \widehat{T}_0) = (\widehat{B}_0^H, \widehat{T}_0^H)$. Then we have in particular an isomorphism of complex tori $\widehat{\eta}_{T_0^H} : \widehat{T}_0^H \to \widehat{T}_0$. There exists an element $\omega \in Z^1(\Gamma, \Omega(\widehat{T}_0, \widehat{G}))$ s.t. $\omega(\sigma)\sigma \circ \widehat{\eta}|_{\widehat{T}_0^H} \circ \sigma^{-1} = \widehat{\eta}|_{\widehat{T}_0^H}$ for all $\sigma \in \Gamma$. Thus we dually obtain an isomorphism of F-tori $\eta : T_0^\omega \to T_0^H$, where T_0^ω denotes the twist of T_0 by ω.

By [Hal93, Lemma 6.1] the map $\widehat{\eta} : \widehat{H} \to \widehat{G}$ can be extended to an L-embedding ${}^L\eta : {}^LH \to {}^LG$ in such a way, that the 1-cocycle

$$I_F \to {}^LH \to \widehat{H}$$

is trivial. We choose such an extension. The extended triple $(H, s, {}^L\eta)$ is then unramified in the sense of [Hal93].

3.1 Review of the construction of DeBacker and Reeder

In this section we want to review the construction from [DR09] of the L-packet on G and its pure inner forms corresponding to a Langlands parameter $\varphi : W_F \to {}^LG$ which is TRSELP in the sense of loc. cit. Our purpose is not to review the details of the construction, but rather to gather the necessary notation and properties needed in the subsequent sections.

Recall that φ is called TRSELP if it is trivial on $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{Cent}(\varphi(I_F), \widehat{G})$ is a maximal torus of $\widehat{G}$, and $Z(\widehat{G})^\Gamma$ is of finite index in $\mathrm{Cent}(\varphi, \widehat{G})$. Up to equivalence we may assume that $\varphi(I_F) \subset \widehat{T}_0$. There is an element $w \in Z^1(\Gamma, \Omega(\widehat{T}_0, \widehat{G}))$ s.t.

$$\mathrm{Ad}(\varphi(\sigma))|_{\widehat{T}_0} \quad = \quad w(\sigma)\sigma, \quad \forall \sigma \in W_F$$

Let T_0^w be the twist of T_0 by w. The ellipticity of φ implies that T_0^w/Z is anisotropic. Put $X = X_*(T_0^w)$. This is a $\mathbb{Z}[\Gamma]$-module, where the Γ-action comes from that on T_0^w. Let $\bar{X}$ be the quotient of X by the coroot-lattice, and X_Γ resp. $\bar{X}_\Gamma$ denote the Γ-coinvaraints in X resp. $\bar{X}$. Let X_w be the preimage of $[X_\Gamma]_{\mathrm{tor}}$ in X. Write C_φ for the component group of the centralizer of φ in $\widehat{G}$. We have

the following diagram

$$\begin{tikzcd}
\mathrm{Irr}(C_\varphi) \arrow[r, equal] & \mathrm{Irr}(\pi_0(\widehat{T_0^w}^\Gamma)) \arrow[r] & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\
X_w \arrow[r] & {[X_\Gamma]_{\mathrm{tor}}} \arrow[u, "\cong"] \arrow[r] \arrow[d, "\cong"', "\mathrm{DR}_{T_0^w}"] & {[\bar{X}_\Gamma]_{\mathrm{tor}}} \arrow[u, "\cong"'] \arrow[d, "\mathrm{DR}_G"', "\cong"] \\
& H^1(\Gamma, T_0^w) \arrow[r] & H^1(\Gamma, G)
\end{tikzcd} \tag{3.1}$$

The bottom square of it is [DR09, Lemma 2.6.1]. The top equality follows from

$$\mathrm{Cent}(\varphi, \widehat{G}) = \widehat{T}_0^{w\vartheta} = \widehat{T_0^w}^\Gamma$$

while the rest is given by the obvious restriction maps.

The map $X_w \to H^1(\Gamma, G)$ in this diagram will be denoted by r. For any $u \in H^1(\Gamma, G)$ we let $[r^{-1}(u)]$ be the image of $r^{-1}(u)$ in $[X_\Gamma]_{\mathrm{tor}}$. The map $X_w \to \mathrm{Irr}(C_\varphi)$ will be denoted by $\lambda \mapsto \rho_\lambda$.

From the Langlands parameter φ DeBacker and Reeder construct (see [DR09, §4]) a Langlands parameter $\varphi_T : W_F \to {}^L T_0^w$ which corresponds to a regular depth-zero character $\theta : T_0^w(F) \to \mathbb{C}^\times$ (both notations θ and χ_φ are used for this character in loc.cit). Moreover, given $\lambda \in X_w$, they construct the following objects

- An element $u_\lambda \in Z^1(\Gamma, G)$ (trivial on inertia). Let $(\psi_\lambda, u_\lambda) : G \to G^\lambda$ be the corresponding twist.
- A maximal torus $T_\lambda \subset G^\lambda$, together with an element $p_\lambda \in G^\lambda(F^u)$ s.t.

  $$\mathrm{Ad}(p_\lambda)\psi_\lambda : T_0^w \to T_\lambda$$

  is an isomorphism of F-tori
- A depth-zero supercuspidal representation π_λ of $G^\lambda(F)$.

Furthermore they show in the proof of [DR09, Thm 4.5.3] that for $\lambda, \mu \in X_w$ one has $\rho_\lambda = \rho_\mu$ if and only if $\psi_\lambda \circ \psi_\mu^{-1} : G^\mu \to G^\lambda$ is a trivial twist and the transfer of π_μ to G^λ with respect to one (hence any) strongly trivial twist equivalent to $\psi_\lambda \circ \psi_\mu^{-1}$ coincides with π_λ. Thus if we put

$$I = \{(G^\lambda, \psi_\lambda, u_\lambda)|\ \lambda \in X_w\}$$

and construct G^I as in Section 2, then for each $\rho \in \mathrm{Irr}(C_\varphi)$ we obtain a conjugation-invariant function $\Theta_{\varphi,\rho}$ on $G^I(F)$ by taking any $\lambda \in X_w$ s.t. $\rho_\lambda = \rho$ and extending the character of π_λ to a conjugation-invariant function on $G^I(F)$.

To simplify their stability calculations, DeBacker and Reeder rigidify their constructions in the following way. In every class of $H^1(\Gamma, G)$ they choose a specific representative $u \in Z^1(\Gamma, G)$, which again gives rise to a twist $(\psi, u) : G \to$

G^u. For each $\lambda \in r^{-1}(u)$ they construct an element $q_\lambda \in G^u(F^u)$ s.t. the maximal torus $S_\lambda = \mathrm{Ad}(q_\lambda)\psi(T_0)$ is defined over F and

$$\mathrm{Ad}(q_\lambda)\psi : T_0^w \to S_\lambda$$

is an isomorphism over F. For any strongly regular semi-simple element $Q \in S_0(F)$ the map

$$\lambda \mapsto \mathrm{Ad}(q_\lambda)\psi\mathrm{Ad}(q_0^{-1})Q$$

is a bijection from $[r^{-1}(u)]$ to a set of representatives for the stable class of Q in $G^u(F)$ ([DR09, Lem. 2.10.1]). In particular, the tori S_λ exhaust the stable class of T_0^w in G^u. It will be important for later to note that $p_0 = q_0 \in G(O_{F^u})$. For every $\rho \in \mathrm{Irr}(C_\varphi)$ mapping to the class of u, they define a representation $\pi_u(\varphi, \rho)$ on $G^u(F)$. It is equal to the transfer of π_λ via any strongly trivial twist $G^\lambda \to G^u$ equivalent to $\psi \circ \psi_\lambda^{-1}$, where λ is any element of $r^{-1}(u)$.

It is clear from the constructions that for any $\lambda \in r^{-1}(u)$, the twist $\psi_\lambda \circ \psi^{-1}$ defines an injection from the conjugacy classes in $G^u(F)$ to the conjugacy classes in $G^I(F)$ whose image consists of those conjugacy classes which meet $G^\mu(F)$ for $\mu \in r^{-1}(u)$. Moreover, this twist identifies the character of $\pi_u(\varphi, \rho)$ with the function $\Theta_{\varphi,\rho}$, where both are viewed as class functions.

The same construction can be applied to a TRSELP $\varphi^H : W_F \to {}^L H$ and the corresponding objects will carry the superscript H.

3.2 The Whittaker character

We extend the chosen pair (T_0, B_0) of G to a splitting $(T_0, B_0, \{X_\alpha\})$ where each simple root vector X_α is chosen so that the homomorphism

$$\mathbb{G}_a \to G$$

determined by it is defined over O_{F^u} and the image of 1 under

$$\mathbb{G}_a(O_{F^u}) \to G(O_{F^u}) \to G(\overline{\mathbb{F}_q})$$

is non-trivial. Such a splitting is called admissible by [Hal93]. Let N denote the unipotent radical of B_0.

Lemma 3.2.1. *There exists an additive character $\psi : F \to \mathbb{C}^\times$ which is non-trivial on O_F but trivial on πO_F, s.t. the representation $\pi_1(\varphi, 1)$ is generic with respect to the character $N(F) \to \mathbb{C}^\times$ determined by ψ and the chosen splitting.*

Proof: The representation $\pi_1(\varphi, 1)$ is the same as the representation π_0 defined in [DR09, §4.5]. By Lemmas 6.2.1 and 6.1.2 in loc. cit. it is generic with respect to a character $N(F) \to \mathbb{C}^\times$ which has depth-zero at o. This character is generic and is thus given by the composition of the F-homomorphism

$$N \to \prod_{\alpha \in \Delta} \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$$

determined by the chosen splitting with an additive character

$$\psi : F \to \mathbb{C}^\times$$

The choice of the simple root vectors X_α ensures that the homomorphism $N \to \mathbb{G}_a$ is in fact defined over O_F and moreover maps $N(O_F)$ surjectively onto $\mathbb{G}_a(O_F)$. The genericity of the character $N(F) \to \mathbb{C}^\times$ now implies that ψ is non-trivial on O_F and trivial on πO_F. $\square$

From now on we fix an additive character $\psi : F \to \mathbb{C}^\times$ as in the above Lemma.

3.3 Definition of the unstable character

For $t \in \mathrm{Cent}(\varphi, \widehat{G})$ we define on $G^I(F)$ the function

$$\Theta_\varphi^t = \sum_{\rho \in \mathrm{Irr}(C_\varphi)} e_\rho \mathrm{tr}\, \rho(t) \Theta_{\varphi,\rho}$$

where for any $\lambda \in X_w$ with $\rho_\lambda = \rho$ we put $e_\rho = e(G^\lambda)$, the latter being the sign defined in [Kot83]. This is the t-unstable character corresponding to the packet $\Pi(\varphi)$ defined in [DR09, §4.5].

We will also define the t-unstable character of the normalized L-packet $\Pi_u(\varphi)$ defined in [DR09, §4.6] for the specific twists $(\psi, u) : G \to G^u$ considered there. This character is

$$\Theta_{\varphi,u}^t := e(G^u) \sum_{\rho \in \mathrm{Irr}(C_\varphi, u)} \mathrm{tr}\, \rho(t) \Theta_{\pi_u(\varphi,\rho)}$$

where $\mathrm{Irr}(C_\varphi, u)$ is the fiber over u of the map $\mathrm{Irr}(C_\varphi) \to H^1(\Gamma, G)$ given in diagram (3.1). We will show in Lemma 5.2.1 that the map $H^1(\Gamma, G) \to \pi_0(Z(\widehat{G})^\Gamma)$ in diagram (3.1) is a particular normalization of the Kottwitz isomorphism, and so the set $\mathrm{Irr}(C_\varphi, u)$ is the set of all irreducible representations of C_φ which transform under $\pi_0(Z(\widehat{G})^\Gamma)$ by the character corresponding to u via the Kottwitz isomorphism.

The restriction of Θ_φ^1 to $G(F)$, which also equals $\Theta_{\varphi,1}^1$, will be denoted by $\mathcal{S}\Theta_\varphi$.

3.4 Statement of the main result

Before stating the main result, we need to impose some mild conditions on the residual characteristic of F. These restrictions come from the papers [DR09] and [Hal93]. To state them, let n_G denote the smallest dimension of a faithful representation of G, and n_H be the corresponding number for H. Let e be the ramification degree of $F/\mathbb{Q}_p$ and e_G be the minimum over the ramification degrees (again over $\mathbb{Q}_p$) of all splitting fields of maximal tori of G. The restrictions we impose are

- $q_F \geq |R(T_0, B_0)|$
- $p \geq (2 + e) \max(n_G, n_H)$
- $p \geq 2 + e_G$

The first two items are imposed in [DR09, §12.4], while the third is imposed in the main result of [Hal93] – Theorem 10.18.

From now on we assume that these restrictions hold.

Let $\varphi^H : W_F \to {}^L H$ be a Langlands parameter for H, then $\varphi = {}^L\eta \circ \varphi^H$ is a Langlands parameter for G. We are interested in the situation in which both φ^H and φ are TRSELP. Then $(H, s, \widehat{\eta})$ is automatically an elliptic endoscopic triple for G. Up to equivalence we may assume that φ^H maps inertia into $\widehat{T}_0^H$, then φ maps inertia into $\widehat{T}_0$ by our choice of $\widehat{\eta}$. There are elements $w \in Z^1(\Gamma, \Omega(\widehat{T}_0, \widehat{G}))$, $w^H \in Z^1(\Gamma, \Omega(\widehat{T}_0^H, \widehat{H}))$ s.t.

$$\begin{aligned} \mathrm{Ad}(\varphi(\sigma))|_{\widehat{T}_0} &= w(\sigma)\sigma, \quad \forall \sigma \in W_F \\ \mathrm{Ad}(\varphi^H(\sigma))|_{\widehat{T}_0^H} &= w^H(\sigma)\sigma, \quad \forall \sigma \in W_F \end{aligned}$$

Let Δ_ψ be the Whittaker normalization [KS99, §5.3] of the absolute transfer factor for (G, H) with respect to the generic character on $N(F)$ determined by ψ and let Δ_ψ^I be its extension to G^I defined in Section 2.2. We will identify the element $s \in Z(\widehat{H})^\Gamma$ with its image in $\widehat{T}_0$ under $\widehat{\eta}$. Then from Section 3.3 we have the functions Θ_φ^s on $G^I(F)$ and $\mathcal{S}\Theta_{\varphi^H}$ on $H(F)$. The main result of this paper is

Theorem 3.4.1. *For any strongly regular semi-simple element $\gamma \in G^I(F)$ the following equality holds*

$$\Theta_\varphi^s(\gamma) = \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_\psi^I(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H)$$

In terms of the normalized L-packets, this statement can be reformulated as follows. Let $(\varphi, u) : G \to G^u$ be a pure inner twist of the type considered in [DR09, §4.6] and let $\Delta_{\psi,u}$ be the normalization of the absolute transfer factor for (G^u, H) corresponding to Δ_ψ as in Section 2.2. Then

Theorem 3.4.2. *For any strongly regular semi-simple element $\gamma \in G^u(F)$ the following equality holds*

$$\Theta_{\varphi,u}^s(\gamma) = \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H) \tag{3.2}$$

4 Endoscopic signs

In this section we only need the notation from the beginning of Section 3. Moreover, it is independent of the restrictions posed on p in Section 3.4. The only restriction we impose on p is $p > 2$, although this again is just for convenience and could be removed.

There are three signs which can be assigned to the pair of groups (G, H) (and some auxiliary choices) and which we need to equate. The first one is

$$\epsilon(G, H) = (-1)^{r_G - r_H}$$

where r_G and r_H are the F-split ranks of G and H. This sign plays an important role in the character formulas of [DR09].

The second sign enters in the normalization of the geometric transfer factors. It is defined relative to an additive character $\psi : F \to \mathbb{C}^\times$ as the local ϵ-factor $\epsilon_L(V,\psi)$ where V is the virtual representation of Γ of degree 0 given by the difference of the Γ-representations $V_G := X^*(T_0) \otimes \mathbb{C}$ and $V_H := X^*(T_0^H) \otimes \mathbb{C}$.

The third appears in Waldspurger's work [Wal95] on the local trace formula for Lie algebras. To construct it, let $\psi : F \to \mathbb{C}^\times$ be an additive character and $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ a non-degenerate, $\mathrm{Ad}(G(F))$-invariant, symmetric bilinear form. With this data, Waldspurger defines in [Wal95, VIII] for a lattice $r \subset \mathfrak{g}(F)$

$$\begin{aligned} I(r) &= \int_r \psi(B(x,x)/2)dx \\ \tilde{r} &= \{x \in \mathfrak{g}(F) | \forall y \in r \; \psi(B(x,y)) = 1\} \end{aligned}$$

and remarks that the function

$$r \mapsto \frac{I(r)}{|I(r)|}$$

is constant when restricted to the set $\{r | \tilde{r} \subset 2r\}$. This constant he then calls $\gamma_\psi(B)$, or $\gamma_\psi(\mathfrak{g})$ when B is understood. Furthermore, in loc. cit. Waldspurger explains how to transfer B to a non-degenerate, $\mathrm{Ad}(H(F))$-invariant, symmetric bilinear form $B_\mathfrak{h}$ on $\mathfrak{h}(F)$, thereby obtaining $\gamma_\psi(B_\mathfrak{h})$. The second sign we are interested in is $\gamma_\psi(B)\gamma_\psi(B_\mathfrak{h})^{-1}$. (The word "sign" is not yet justified here, all we know is that both constants and hence their quotient are eight roots of unity. We will see however that in our case the quotient is a sign.)

We extend the bilinear form B to a symmetric bilinear form $\mathfrak{g}(\overline{F}) \times \mathfrak{g}(\overline{F}) \to \overline{F}$ in the obvious way and denote it by the same letter. As remarked in loc.cit., this extension is $\mathrm{Ad}(G(\overline{F})) \rtimes \Gamma$-invariant. It is clear that if $V \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ defined over some extension E of F, then the restriction of B to V defines a symmetric bilinear form $V(E) \times V(E) \to E$.

The purpose of this section is to prove the following

Proposition 4.0.3. *Let $\psi : F \to \mathbb{C}^\times$ be an additive character which is non-trivial on O_F and trivial on πO_F. Let B be a "good" bilinear form in the sense of [DR09, A.1]. Then*

$$\epsilon_L(V,\psi) = \epsilon(G,H) = \gamma_\psi(B)\gamma_\psi(B_\mathfrak{h})^{-1}$$

The proof is contained in the following lemmas.

Remark: We would like to point out that the second of these equalities is also proved in [KV2]. The proof given here is different from the one in loc. cit. and establishes a connection between the above signs and the number of symmetric orbits of Γ in $R(T^H, G)$. This number is an important invariant in endoscopy and thus the following lemmas may be of independent interest.

Lemma 4.0.4.

$$\epsilon(G,H) = \det(\omega)$$

Proof: A similar argument is given in the proof of [DR09, Lemma 12.3.5], but we will present it here since our situation and notation are different. ϑ is a finite-order automorphism of the real vector space $X^*(T_0) \otimes \mathbb{R}$ and hence is

diagonalizable over $\mathbb{C}$ with eigenvalues roots of unity, and all non-real eigenvalues come in conjugate pairs. Thus $\det(\vartheta) = (-1)^{\dim(V_G)-\dim(V_G^{\vartheta})}$. In the same way $\det(\omega\vartheta) = (-1)^{\dim(V_H)-\dim(V_H^{\omega\vartheta})}$. But

$$\epsilon(G,H) = (-1)^{\dim(V_G^{\Gamma})-\dim(V_H^{\Gamma})} = (-1)^{\dim(V_G^{\vartheta})-\dim(V_H^{\omega\vartheta})} = \det(\omega)$$

□

Lemma 4.0.5.

$$\epsilon_L(V,\psi) = \det(\omega)$$

Proof: The Γ representations V_G and V_H are unramified. Applying [Tat77, 3.4.6] and noting that the isomorphism of local class field theory used in loc. cit. is normalized so that Fi corresponds to π, we obtain

$$\begin{aligned} \epsilon_L(V_G - V_H, \psi) &= \det V_G(\mathrm{Fi}^{-1}) \det V_H(\mathrm{Fi}^{-1})^{-1} \\ &= \left[\frac{\det(\vartheta)}{\det(\omega\vartheta)}\right]^{-1} \\ &= \det(\omega) \end{aligned}$$

□

These two lemmas complete the proof of the first equality in Proposition 4.0.3. To continue with the second equality, we need to recall some notions from [LS87]. Let T be a maximal torus of G, and $\mathcal{O}$ be a Γ-orbit in $R(T,G)$, the set of roots of T in G. Then $-\mathcal{O}$ is also a Γ-orbit in $R(T,G)$ and either $\mathcal{O} = -\mathcal{O}$, in which case $\mathcal{O}$ is called a *symmetric* orbit, or $\mathcal{O} \cap -\mathcal{O} = \emptyset$, in which case $\mathcal{O}$ is called an *asymmetric* orbit. For $\alpha \in R(T,G)$ let Γ_α be the stabilizer of α and $\Gamma_{\pm\alpha}$ be the stabilizer of the set $\{\alpha, -\alpha\}$. Let F_α and $F_{\pm\alpha}$ be the fixed fields of Γ_α and $\Gamma_{\pm\alpha}$ in $\overline{F}$. Then $[\Gamma_\alpha, \Gamma_{\pm\alpha}]$ equals 2 if the orbit of α is symmetric and 1 if it is asymmetric. If T is unramified, then both F_α and $F_{\pm\alpha}$ lie in F^u.

For any Γ-invariant subset $S \subset R(T,G)$ we put

$$\mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

This is clearly a vector subspace of $\mathfrak{g}$ defined over F.

Lemma 4.0.6. *Let T be a maximal torus of G stably conjugate to T_0^ω. Then*

$$\gamma_\psi(B)\gamma_\psi(B_{\mathfrak{h}})^{-1} = \prod_{\mathcal{O}} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}}(F)})$$

where $\mathcal{O}$ runs over the set of symmetric orbits of Γ in $R(T,G)$.

Proof: We consider the root decomposition of $\mathfrak{g}$ relative to T:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(T,G)} \mathfrak{g}_\alpha$$

If we put $\mathfrak{g}_0 = \mathfrak{t}$ then the invariance of B implies that for all $\alpha, \beta \in R(T,G) \cup \{0\}$ such that $\alpha \neq -\beta$ the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ of $\mathfrak{g}$ are orthogonal with respect to

B. This means that if $\mathcal{O}_1, ..., \mathcal{O}_k$ are the orbits in $R(T, G)$ of the group $\Gamma \times \{\pm 1\}$, where $\{\pm 1\}$ acts by scalar multiplication, then

$$\mathfrak{g}(F) = \mathfrak{t}(F) \oplus \bigoplus_{i=1}^{k} \mathfrak{g}_{\mathcal{O}_i}(F)$$

is an orthogonal decomposition of $\mathfrak{g}(F)$. Thus $\gamma_\psi(B)$ factors as

$$\gamma_\psi(B) = \gamma_\psi(B|_{\mathfrak{t}(F)}) \prod_{i=1}^{k} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$$

Consider one of the orbits $\mathcal{O}_i$. Either Γ acts transitively on it, in which case it is a symmetric Γ-orbit, or it decomposes as a disjoint union of two asymmetric Γ-orbits. We assume that the latter is the case, and write $\mathcal{O}_i = \mathcal{O}'_i \sqcup -\mathcal{O}'_i$ where $\mathcal{O}'_i$ is one of the two Γ orbits in $\mathcal{O}_i$. Then $\mathfrak{g}_{\mathcal{O}_i} = \mathfrak{g}_{\mathcal{O}'_i} \oplus \mathfrak{g}_{-\mathcal{O}'_i}$ is a decomposition over F as a direct sum of isotropic spaces. Let $r_+ \subset \mathfrak{g}_{\mathcal{O}'_i}(F)$ and $r_- \subset \mathfrak{g}_{-\mathcal{O}'_i}(F)$ be large enough lattices. Then $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$ is by definition the complex sign of

$$\begin{aligned} &\int_{r_+ \oplus r_-} \psi(B(x+y, x+y)/2) d(x,y) \\ = \quad &\int_{r_+} \int_{r_-} \psi(B(x,y)) dx dy \end{aligned}$$

For each $x \in r_+$ the map $y \mapsto \psi(B(x,y))$ is a character of the additive group r_-. Thus if r_+^0 is the subgroup of r_+ consisting of all x s.t. this character is trivial, the above integral is equal to the positive real constant $\mathrm{vol}(r_+^0, dx)\mathrm{vol}(r_-, dy)$. This shows $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)}) = 1$ and we conclude that

$$\gamma_\psi(B) = \gamma_\psi(B|_{\mathfrak{t}(F)}) \prod_{\mathcal{O}} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$$

where $\mathcal{O}$ runs over the set of symmetric Γ-orbits in $R(T, G)$.

We can apply the same reasoning to the Lie algebra $\mathfrak{h}$ with the bilinear form $B_\mathfrak{h}$ and the torus T_0^H. Since T_0^H is contained in a Borel defined over F, there are no symmetric orbits of Γ in $R(T_0^H, H)$ and we conclude

$$\gamma_\psi(B_\mathfrak{h}) = \gamma_\psi(B_\mathfrak{h}|_{\mathfrak{t}_0^H(F)})$$

But we have chosen the torus T so that there exists an admissible isomorphism $T_0^H \to T$ over F, and the bilinear form $B_\mathfrak{h}$ is constructed so that the differential of this admissible isomorphism identifies $B_\mathfrak{h}|_{\mathfrak{t}_0^H(F)}$ with $B|_{\mathfrak{t}(F)}$. Thus

$$\gamma_\psi(B_\mathfrak{h}) = \gamma_\psi(B|_{\mathfrak{t}(F)})$$

and the lemma now follows. □

Lemma 4.0.7. *Let $\mathcal{O}$ be a symmetric orbit of Γ in $R(T, G)$. Then*

$$\gamma_\psi(B|_{\mathfrak{g}_\mathcal{O}(F)}) = -1$$

Proof: Choose $\alpha \in \mathcal{O}$ and $\sigma_\alpha \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$. We can choose a non-zero $E \in \mathfrak{g}_\alpha(F_\alpha) \cap [\mathfrak{g}_o \smallsetminus \mathfrak{g}_{o^+}]$ and then we have $\sigma_\alpha(E) \in \mathfrak{g}_{-\alpha}(F_\alpha) \cap [\mathfrak{g}_o \smallsetminus \mathfrak{g}_{o^+}]$. Then by [DR09, §A.1]

$$B(E, \sigma_\alpha(E)) \in O_{F_{\pm\alpha}}^\times$$

The map

$$\varphi : F_\alpha \to \mathfrak{g}_{\mathcal{O}}(F), \qquad \lambda \mapsto \sum_{\sigma\in\Gamma/\Gamma_\alpha} \sigma(\lambda E)$$

is an isomorphism of F-vector spaces and clearly $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}}(F)}) = \gamma_\psi(\varphi^* B)$. To compute the bilinear form $\varphi^* B : F_\alpha \times F_\alpha \to F$ we notice that if $\sigma_1, ..., \sigma_k$ are representatives for $\Gamma/\Gamma_{\pm\alpha}$, then

$$\mathfrak{g}_{\mathcal{O}} = \bigoplus_{i=1}^{k} (\mathfrak{g}_{\sigma_i(\alpha)} \oplus \mathfrak{g}_{\sigma_i(-\alpha)})$$

is an orthogonal sum of hyperbolic planes. Then a direct computation shows that

$$\varphi^* B(\lambda, \mu) = \mathrm{tr}_{\,F_{\pm\alpha}/F}\Big([\lambda\sigma_\alpha(\mu) + \mu\sigma_\alpha(\lambda)]B(E, \sigma_\alpha(E))\Big)$$

If we put

$$\begin{aligned} \psi'(x) &= \mathrm{tr}_{\,F_{\pm\alpha}/F}(B(E, \sigma_\alpha(E))x) \\ B'(\mu, \lambda) &= \lambda\sigma_\alpha(\mu) + \mu\sigma_\alpha(\lambda) \end{aligned}$$

then $\psi' : F_{\pm\alpha} \to \mathbb{C}^\times$ is an additive character and $B' : F_\alpha \times F_\alpha \to F_{\pm\alpha}$ is a non-degenerate $F_{\pm\alpha}$-bilinear form, and clearly $\gamma_\psi(\varphi^* B) = \gamma_{\psi'}(B')$.

We will now compute $\gamma_{\psi'}(B')$.

First we claim that ψ' is non-trivial on $O_{F_{\pm\alpha}}$ but trivial on $\pi O_{F_{\pm\alpha}}$. To see this, note that $\mathrm{tr}_{\,F_{\pm\alpha}/F}$ induces for each $i \in \mathbb{Z}$ a homomorphism of additive groups $\pi^i O_{F_{\pm\alpha}} \to \pi^i O_F$ which fits into the diagram

$$\begin{array}{ccc} \pi^i O_{F_{\pm\alpha}} & \longrightarrow & \pi^i O_F \\ \Big\downarrow{\scriptstyle \text{mod } \pi^{i+1}} & & \Big\downarrow{\scriptstyle \text{mod } \pi^{i+1}} \\ k_{F_{\pm\alpha}} & \xrightarrow{\mathrm{tr}} & k_F \end{array}$$

and thus $\mathrm{tr}_{\,F_{\pm\alpha}/F} : O_{F_{\pm\alpha}} \to O_F$ is surjective ([Ser79, V.§1.Lemma 2]). This together with $B(E, \sigma_\alpha(E)) \in O^\times_{F_{\pm\alpha}}$ implies the claim about ψ'.

Next we compute the dual of the $O_{F_{\pm\alpha}}$-lattice O_{F_α} with respect to $\psi' \circ B'$.

$$\begin{aligned} & \{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : \psi'(B(x, y)) = 1\} \\ = \; & \{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : B(x, y) \in \pi O_{F_{\pm\alpha}}\} \\ = \; & \pi\{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : x\sigma_\alpha(y) + y\sigma_\alpha(x) \in O_{F_{\pm\alpha}}\} \\ = \; & \pi\{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : xy + \sigma_\alpha(y)\sigma_\alpha(x) \in O_{F_{\pm\alpha}}\} \end{aligned}$$

Thus we are looking for π times the dual of O_{F_α} with respect to the bilinear form $(x, y) \mapsto \mathrm{tr}_{\,F_\alpha/F_{\pm\alpha}}(xy)$. This dual is the codifferent of $F_\alpha/F_{\pm\alpha}$, which equals O_{F_α} since $F_\alpha/F_{\pm\alpha}$ is an unramified extension.

We conclude that the lattice O_{F_α} has the property that it contains its dual with respect to $\psi' \circ B'$. Since we are imposing the restriction $p > 2$ and thus $O_{F_\alpha} = 2O_{F_\alpha}$. Then by definition, $\gamma_{\psi'}(B')$ is the complex sign of

$$I := \int_{O_{F_\alpha}} \psi'(N(x))dx$$

where $N : F_\alpha \to F_{\pm\alpha}$ is the norm map and dx is a Haar measure on the additive group F_α. Let $(\xi_k)_{k\in k_{F_\alpha}}$ be a set of representatives for $O_{F_\alpha}/\pi O_{F_\alpha}$. Then

$$I = \sum_{k\in k_{F_\alpha}} \int_{\pi O_{F_\alpha}} \psi'(N(\xi_k + x))dx$$

One computes immediately that $\psi'(N(\xi_k + x)) = \psi'(N(\xi_k))$ for all $x \in \pi O_{F_\alpha}$ since ψ' is trivial on $\pi O_{F_{\pm\alpha}}$. This leads to

$$I = \mathrm{vol}(\pi O_{F_\alpha}, dx) \sum_{k\in k_{F_\alpha}} \psi'(N(\xi_k))$$

The restriction of ψ' to $O_{F_{\pm\alpha}}$ factors through the natural projection $O_{F_{\pm\alpha}} \to k_{F_{\pm\alpha}}$, and the composition of N with this projection factors through the projection $O_{F_\alpha} \to k_{F_\alpha}$ and induces the norm map associated to the extension $k_{F_\alpha}/k_{F_{\pm\alpha}}$, which we also call N. Thus

$$\begin{aligned} I &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \sum_{k\in k_{F_\alpha}} \psi'(N(k)) \\ &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[1 + \sum_{k\in k_{F_\alpha}^\times} \psi'(N(k))\right] \end{aligned}$$

Now $N : k_{F_\alpha}^\times \to k_{F_{\pm\alpha}}^\times$ is a surjective homomorphism, the cardinality of whose fibers we will call A. Then

$$\begin{aligned} I &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[1 + A \sum_{k\in k_{F_{\pm\alpha}}^\times} \psi'(k)\right] \\ &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[-(A-1) + A \sum_{k\in k_{F_{\pm\alpha}}} \psi'(k)\right] \\ &= -(A-1)\mathrm{vol}(\pi O_{F_\alpha}, dx) \end{aligned}$$

since ψ' is a non-trivial character on the additive group $k_{F_{\pm\alpha}}$. We conclude that I is a negative real number, and the lemma follows. □

Lemma 4.0.8.

$$\det(\omega) = (-1)^N$$

where N is the number of symmetric orbits of Γ in $R(T, G)$.

Proof: We choose a $g \in G(\overline{F})$ s.t. $\mathrm{Ad}(g) : T_0^\omega \to T$ is an isomorphism over F and use it to regard ω and ϑ as automorphisms of $R(T, G)$. Moreover put $B = \mathrm{Ad}(g)B_0$ and write $\alpha > 0$ if $\alpha \in R(T, B)$. Let

$$\begin{aligned} S &= \{\alpha \in R(T, G)|\alpha > 0 \wedge \omega\alpha < 0\} \\ S' &= \{\alpha \in R(T, G)|\alpha > 0 \wedge \omega\vartheta\alpha < 0\} \end{aligned}$$

Since ϑ preserves the set of positive roots in $R(T, G)$, it induces a bijection $S' \to S$. Thus

$$\det(\omega) = (-1)^{|S|} = (-1)^{|S'|}$$

<u>Claim 1:</u> The cardinality of S' is congruent mod 2 to the cardinality of the intersection of S' with the union of the symmetric orbits of Γ in $R(T, G)$.

Put $T = \omega\vartheta$ for short. Then Γ acts on $R(T, G)$ via the cyclic group $< T >$. Let $\mathcal{O}$ be an orbit. We claim that the sets

$$\begin{aligned} \mathcal{O}_+ &= \{\alpha \in \mathcal{O} |\ \alpha > 0 \wedge T\alpha < 0\} \\ \mathcal{O}_- &= \{\alpha \in \mathcal{O} |\ \alpha < 0 \wedge T\alpha > 0\} \end{aligned}$$

have the same cardinality. To see this, consider the directed graph in the vector space $X^*(T_{\mathrm{ad}}) \otimes \mathbb{R}$ whose vertices are given by $\mathcal{O}$ and whose edges are given by

$$\{(\alpha, T\alpha) |\ \alpha \in \mathcal{O}\}$$

Then $\mathcal{O}_+$ is in bijection with the set of edges which start in the positive half space of $X^*(T_{\mathrm{ad}}) \otimes \mathbb{R}$ and end in the negative, while $\mathcal{O}_-$ is in bijection with the set of edges which start in the negative half space and end in the positive. But our graph is a closed loop, so these sets must have the same cardinality.

If $\mathcal{O}$ is an asymmetric orbit, then $-\mathcal{O}$ is also one and is disjoint from $\mathcal{O}$, and multiplication by -1 gives a bijection $\mathcal{O}_- \to [-\mathcal{O}]_+$. We conclude that $S' \cap (\mathcal{O} \cup -\mathcal{O})$ has an even cardinality. This proves Claim 1.

Claim 2: Let $\mathcal{O}$ be a symmetric orbit. Then its intersection with S' has an odd cardinality.

The group $< T >$ acts on $\mathcal{O}/\{\pm 1\}$ and all elements of the latter set are of the form $\{\alpha, -\alpha\}$ with $\alpha \in \mathcal{O}$. We choose an element $A \in \mathcal{O}/\{\pm 1\}$, and let $n = |\mathcal{O}|/2 - 1$. Then $A, TA, ..., T^n A$ enumerates $\mathcal{O}/\{\pm 1\}$. For each $0 \leq i \leq n$ let α_i be the positive member of $T^i A$. Then for each such i one of two cases occurs: either $T\alpha_i = -\alpha_{i+1}$ and $T(-\alpha_i) = \alpha_{i+1}$, or $T\alpha_i = \alpha_{i+1}$ and $T(-\alpha_i) = -\alpha_{i+1}$ (where we adopt the convention $\alpha_{n+1} = \alpha_0$). The cardinality of $S' \cap \mathcal{O}$ is the number of $0 \leq i \leq n$ for which the first case occurs. Now let M be the number of $0 \leq i < n$ for which the first case occurs (note the sharp inequality!). If M is even, then $T^n\alpha_0 = \alpha_n$ and thus $T\alpha_n$ must equal $-\alpha_0$, for otherwise the set $\{\alpha_0, T\alpha_0, ..., T^n\alpha_0\}$ will be a T-invariant subset of $\mathcal{O}$, which is impossible. Thus $|S' \cap \mathcal{O}| = M + 1$ is an odd number. If conversely M is odd, then $T^n\alpha_0 = -\alpha_n$ and by the same reasoning $T(-\alpha_n) = (-\alpha_0)$. It follows then that $|S' \cap \mathcal{O}| = M$, again an odd number. This proves Claim 2.

The two claims together imply that $(-1)^{|S'|} = (-1)^N$ and this finishes the lemma. □

The second equality in Proposition 4.0.3 now follows from these lemmas.

5 A formula for the unstable character

The purpose of this section is to establish a reduction formula, similar to the ones in [DR09, §9,§10], for $\Theta^t_{\rho,u}$. Before we can do so, we need some cohomological facts.

5.1 Cohomological lemmas I

We begin by recalling some well-known basic facts about Tate-Nakayama duality as used in endoscopy. For this, we will deviate from the notation established so far in order to make the statements in their natural generality. Let

E/F be a finite extension of local fields of characteristic 0, $\Gamma = \mathrm{Gal}(E/F)$, $u_{E/F} \in H^2(\Gamma, E^\times)$ the canonical class of the extension E/F, T a torus over F which splits over E, and $\widehat{T}$ its dual complex torus.

Lemma 5.1.1. *We have the exact sequences*

$$1 \longrightarrow (\widehat{T}^\Gamma)^\circ \longrightarrow \widehat{T}^\Gamma \longrightarrow H^1(\Gamma, X_*(\widehat{T})) \longrightarrow 0$$

$$0 \longrightarrow X^*(\widehat{T}/\widehat{T}^\Gamma) \longrightarrow X^*(\widehat{T}/(\widehat{T}^\Gamma)^\circ) \longrightarrow H_T^{-1}(\Gamma, X^*(\widehat{T})) \longrightarrow 0$$

Proof: For the first one, tensor the exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e^{2\pi iz}} \mathbb{C}^\times \to 1$$

with $X_*(T)$ and take Γ-invariants, noting that the image of $[X_*(\widehat{T}) \otimes \mathbb{C}]^\Gamma = \mathrm{Lie}(\widehat{T})^\Gamma$ in $\widehat{T}^\Gamma$ under the exponential map is $(\widehat{T}^\Gamma)^\circ$.

For the second one, observe that an element of $X^*(\widehat{T})$ is in the kernel of the norm map precisely when it is trivial on $(\widehat{T}^\Gamma)^\circ$ and in the augmentation submodule precisely when it is trivial on $\widehat{T}^\Gamma$. $\square$

Lemma 5.1.2. *The following three pairings*

$$H^1(\Gamma, X_*(\widehat{T})) \otimes H_T^{-1}(\Gamma, X^*(\widehat{T})) \to \mathbb{C}^\times$$

are equal.

1. *The pairing induced by the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$ via the above sequences.*

2. $$\begin{matrix} H^1(\Gamma, X_*(\widehat{T})) \\ \otimes \\ H_T^{-1}(\Gamma, X^*(\widehat{T})) \end{matrix} \xrightarrow{\ \cup\ } H_T^0(\Gamma, \mathbb{Z}) = \mathbb{Z}/|\Gamma|\mathbb{Z} \xrightarrow{\cdot|\Gamma|^{-1}} \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi iz}} \mathbb{C}^\times$$

3. $$\begin{matrix} H^1(\Gamma, X_*(\widehat{T})) = H^1(\Gamma, X^*(T)) \\ \otimes \\ H_T^{-1}(\Gamma, X_*(T)) \xrightarrow{\cup u_{E/F}} H^1(\Gamma, T) \end{matrix} \xrightarrow{\ \cup\ } H^2(\Gamma, E^\times) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi iz}} \mathbb{C}^\times$$

Proof: The equality of the pairings in 2. and 3. is an immediate consequence of local class field theory, more precisely of the following commutative square.

$$\begin{array}{ccc} H_T^2(\Gamma, E^\times) & \xleftarrow{\cup u_{E/F}} & H_T^0(\Gamma, \mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{inv}} & & \| \\ |\Gamma|^{-1}\mathbb{Z}/\mathbb{Z} & \xleftarrow{\cdot|\Gamma|^{-1}} & \mathbb{Z}/|\Gamma|\mathbb{Z} \end{array}$$

In order to relate pairings 1. and 2. take $t \in \widehat{T}^\Gamma$ and $\varphi \in X^*(\widehat{T}/(\widehat{T}^\Gamma)^\circ)$. Choose $z \in \mathrm{Lie}(\widehat{T}) = X_*(\widehat{T}) \otimes \mathbb{C}$ mapping to t under the exponential map. Then the image of t in $H^1(\Gamma, X_*(\widehat{T}))$ is represented by the cocycle $\tau \mapsto \tau z - z$. Now using the appropriate cup product formula and denoting the canonical pairing $X^*(\widehat{T}) \otimes X_*(\widehat{T}) \to \mathbb{Z}$ by $\langle\rangle$ we compute

$$\begin{aligned}(\tau z - z) \cup \varphi &= \sum_{\tau\in\Gamma} \langle \tau\varphi, \tau z - z\rangle \\ &= |\Gamma|\langle\varphi, z\rangle\end{aligned}$$

Note that we have used that φ is in the kernel of the norm map. It follows that

$$\exp(2\pi i|\Gamma|^{-1}(\tau z - z) \cup \varphi) = \exp(2\pi i\langle\varphi, z\rangle) = \langle\varphi, t\rangle$$

□

Lemma 5.1.3. *Assume that E/F is either an unramified extension of p-adic fields, or $\mathbb{C}/\mathbb{R}$. In the p-adic case, let $\pi \in E^\times$ be a uniformizer and $\sigma \in \Gamma$ be the Frobenius element. In the real case, let $\pi = -1$ and $\sigma \in \Gamma$ be complex conjugation. Then the map*

$$[\lambda] \mapsto \lambda(\pi)$$

induces the same isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} = H_T^{-1}(\Gamma, X_*(T)) \to H^1(\Gamma, T)$$

as the isomorphism given by $\cup u_{E/F}$. Here we regard $\lambda(\pi) \in T(E)$ as the class in $H^1(\Gamma, T)$ represented by the unique element $z \in Z^1(\Gamma, T)$ s.t. $z(\sigma) = \lambda(\pi)$.

Proof: By definition, $H_T^{-1}(\Gamma, X_*(T)) = \mathrm{Ker}(N : X_*(T) \to X_*(T))/IX_*(T)$ where N is the norm map and $I \subset \mathbb{Z}[\Gamma]$ is the augmentation ideal. If $\lambda \in X_*(T)$ is torsion modulo $IX_*(T)$, then some multiple of it is killed by N, and since $X_*(T)$ is torsion-free this means that λ itself is killed by N. Thus

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \subset H_T^{-1}(\Gamma, X_*(T))$$

The converse inclusion follows from the finiteness of $H_T^{-1}(\Gamma, X_*(T))$. This justifies the first equality.

It is well known from local class field theory that the fundamental class of E/F is represented by the 2-cocycle

$$(\sigma^a, \sigma^b) \mapsto \begin{cases} 1 & , 0 \le a + b < |\Gamma| \\ \pi & , \text{else} \end{cases}$$

If $\lambda \in X_*(T)$ is torsion modulo $IX_*(T)$, then applying the appropriate cup-product formula one sees

$$([\lambda] \cup u_{E/F})(\sigma) = \sum_{i=0}^{|\Gamma|-1} \sigma^{i+1}\lambda(\pi^{\mathrm{char}_{\{i+1\ge|\Gamma|\}}}) = \lambda(\pi)$$

□

This isomorphism is sometimes called the Tate-Nakayama isomorphism. We will denote it by TN. In the case that E/F is an unramified extension of p-adic fields, DeBacker and Reeder construct in [DR09, Cor 2.4.3] another isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to H^1(\Gamma, T)$$

We will call this isomorphism DR. It turns out that these two isomorphisms are almost identical, namely

Lemma 5.1.4. *The following diagram commutes*

$$\begin{tikzcd}
{[X_*(T)_\Gamma]_{\mathrm{tor}}} \arrow[rr, "\lambda \mapsto -\lambda"] \arrow[dr, "\mathrm{DR}"'] & & {[X_*(T)_\Gamma]_{\mathrm{tor}}} \arrow[dl, "\mathrm{TN}"] \\
& H^1(\Gamma, T) &
\end{tikzcd}$$

Proof: By construction, DR sends $[\lambda] \in [X_*(T)_\Gamma]_{\mathrm{tor}}$ to the class in $H^1(\Gamma, T)$ of the unique cocycle z whose value at Fi equals $t_\lambda = \lambda(\pi)$, while TN sends $[\lambda]$ to the class in $H^1(\Gamma, T)$ of the unique cocycle z' whose value at $\sigma = \mathrm{Fi}^{-1}$ equals $\lambda(\pi)$. But

$$z(\sigma) = \sigma(z(\mathrm{Fi})^{-1}) = \sigma(\lambda(\pi)^{-1}) = \sigma(\lambda)^{-1}(\pi)$$

Since λ and $\sigma(\lambda)$ give rise to the same element of $X_*(T)_\Gamma$, the lemma follows. $\square$

5.2 Cohomological lemmas II

We now return to the previously established notation. Recall the diagram (3.1).

We call a_G the composition

$$H^1(\Gamma, G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$$

of the right vertical isomorphisms in this diagram. In [Kot86, Thm 1.2] Kottwitz defines another isomorphism

$$H^1(\Gamma, G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$$

which he calls α_G. This isomorphism can be normalized in two different ways, and the two normalizations differ by a sign.

Lemma 5.2.1. *Depending on the normalization of α_G, one has*

$$a_G = \pm \alpha_G$$

Proof: Assume first that $G = T$ is a torus. One normalization of the isomorphism α_G is then given by the composition

$$H^1(\Gamma, T) \to \mathrm{Irr}(H^1(\Gamma, X^*(T))) \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

where the first map arises via the cup product pairing

$$H^1(\Gamma, X^*(T)) \otimes H^1(\Gamma, T) \to \mathbb{C}^\times$$

and the second map is the dual of the isomorphism $\pi_0(\widehat{T}^\Gamma) \to H^1(\Gamma, X_*(\widehat{T}))$ of Lemma 5.1.1.

Thus if we precompose α_G by TN then by Lemma 5.1.2 the resulting isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

will be given by the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$.

On the other hand if we precompose a_G by TN then by Lemma 5.1.4 the resulting isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

will be given by the negative of the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$.

This proves that in the case $G = T$ with our normalization of α_G we have $a_G = -\alpha_G$. For the general case let $T \subset G$ be an elliptic maximal torus and consider the commutative diagrams

$$\begin{tikzcd} \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma)) \arrow[r] & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\ H^1(\Gamma,T) \arrow[u, "a_T"] \arrow[r] & H^1(\Gamma,G) \arrow[u, "a_G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma)) \arrow[r] & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\ H^1(\Gamma,T) \arrow[u, "-\alpha_T"] \arrow[r] & H^1(\Gamma,G) \arrow[u, "-\alpha_G"'] \end{tikzcd}$$

The fact that the right diagram commutes is part of the statement of [Kot86, Thm 1.2], while for the left diagram it follows from the construction. We just proved that the left vertical arrows in the two diagrams coincide. But since T is elliptic, the bottom horizontal maps are surjective by [Kot86, Lemma 10.1]. Thus the right vertical maps in the two diagrams must also coincide. $\square$

Lemma 5.2.2. *Let $Q_0 \in \mathrm{Lie}(S_0)(F)$ be a regular semi-simple element and $\lambda \in r^{-1}(u)$. Put $Q_\lambda := \mathrm{Ad}(q_\lambda q_0^{-1})Q_0$. Then*

1. *$Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$.*
2. *The image of λ under the map*

$$[X_\Gamma]_{\mathrm{tor}} \xrightarrow{\mathrm{DR}} H^1(\Gamma, T_0^w) \xrightarrow{\mathrm{Ad}(q_0)} H^1(\Gamma, S_0)$$

   equals $\mathrm{inv}(Q_0, Q_\lambda)$.
3. *The map $\lambda \mapsto Q_\lambda$ establishes a bijection from $[r^{-1}(u)]$ to a set of representatives for the conjugacy classes of elements in $\mathrm{Lie}(G^u)(F)$ stably conjugate to Q_0.*
4. *Let $t \in [\widehat{T}_0^w]^\Gamma$ and t_{q_0} be its image under the dual of $\mathrm{Ad}(q_0^{-1}) : S_0 \to T_0^w$. Then*

$$\langle \mathrm{inv}(Q_0, Q_\lambda), t_{q_0} \rangle^{-1} = \lambda(t)$$

Proof: Recall from [DR09, §2.8] the equations

$$q_\lambda^{-1} u \mathrm{Fi}_G(q_\lambda) = t_\lambda \dot{w} \qquad\qquad q_0^{-1}\mathrm{Fi}_G(q_0) = \dot{w}$$

where $t_\lambda = \lambda(\pi)$. The inner twist ψ is unramified, so $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F^u)$. To prove that $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$ it is enough to show that it is fixed by $\mathrm{Fi}_{G^u} = \mathrm{Ad}(u) \circ \mathrm{Fi}_G$.

$$\begin{aligned} \mathrm{Ad}(u)\mathrm{Fi}_G(Q_\lambda) &= \mathrm{Ad}(u)\mathrm{Fi}_G\mathrm{Ad}(q_\lambda q_0^{-1})Q_0 \\ &= \mathrm{Ad}(u\mathrm{Fi}_G(q_\lambda q_0^{-1}))Q_0 \\ &= \mathrm{Ad}(q_\lambda \dot{w} \mathrm{Fi}_G(q_0^{-1}))Q_0 \\ &= \mathrm{Ad}(q_\lambda \dot{w}\dot{w}^{-1} q_0^{-1})Q_0 \\ &= Q_\lambda \end{aligned}$$

This proves the first assertion.

By construction the element $\mathrm{inv}(Q_0, Q_\lambda)$ is given by the cocycle

$$\sigma \mapsto q_0 q_\lambda^{-1} u \sigma(q_\lambda q_0^{-1})$$

We compute the value of this cocycle at Fi

$$\begin{aligned} q_0 q_\lambda^{-1} u \mathrm{Fi}_G(q_\lambda q_0^{-1}) &= q_0 t_\lambda \dot{w} \mathrm{Fi}_G(q_0^{-1}) \\ &= \mathrm{Ad}(q_0)(t_\lambda \dot{w}(q_0^{-1}\mathrm{Fi}_G(q_0))^{-1}) \\ &= \mathrm{Ad}(q_0)(t_\lambda) \end{aligned}$$

This proves the second assertion.

The third assertion follows immediately from the second and Lemma 2.1.5 (or rather from its Lie-algebra analog, which is proved in the exact same way).

Finally, by functoriality of the Tate-Nakayama pairing we have

$$\langle \mathrm{inv}(Q_0, Q_\lambda), t_{q_0} \rangle^{-1} = \langle \mathrm{Ad}(q_0^{-1}) \mathrm{inv}(Q_0, Q_\lambda), t \rangle^{-1}$$

By the second assertion and Lemma 5.1.4 the element $\mathrm{Ad}(q_0^{-1})(\mathrm{inv}(Q_0, Q_\lambda))^{-1}$ of $H^1(\Gamma, T_0^w)$ is the image of λ under the Tate-Nakayama isomorphism

$$H_T^{-1}(\Gamma, X_*(T_0^w)) \to H^1(\Gamma, T_0^w)$$

and hence by Lemma 5.1.2 we have

$$\langle \mathrm{Ad}(q_0^{-1}) \mathrm{inv}(Q_0, Q_\lambda)^{-1}, t \rangle = \lambda(t)$$

□

5.3 A reduction formula for the unstable character

We now return to the computation of $\Theta_{\rho,u}^t$.

The map

$$[X_\Gamma]_{\mathrm{tor}} \to \mathrm{Irr}(C_\varphi), \qquad \lambda \mapsto \rho_\lambda$$

identifies $[r^{-1}(u)]$ with $\mathrm{Irr}(C_\varphi, u)$. Since it is given simply by restriction of characters, we have $\mathrm{tr}\,\rho_\lambda(t) = \lambda(t)$. Moreover $e(G^u) = \epsilon(G, G^u)$, so

$$\Theta_{\rho,u}^t = \epsilon(G, G^u) \sum_{\lambda \in [r^{-1}(u)]} \lambda(t) \Theta_{\pi_u(\varphi, \rho_\lambda)}$$

Our first goal is to use the results of [DR09, §9,§10] to derive a formula for $\Theta_{\pi_u(\varphi,\rho_\lambda)}$ which is suitable for our purposes. Recall that there is a depth-zero character $\theta : T_0^w(F) \to \mathbb{C}^\times$ determined by the Langlands parameter φ.

Lemma 5.3.1. *Let $\lambda \in r^{-1}(u)$, $\theta_\lambda = \mathrm{Ad}(q_\lambda)_*\theta$, and $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$ be any fixed regular semi-simple element. Then for any $\gamma \in G_{\mathrm{sr}}^u(F)_0$ and any $z \in Z(F)$ we have*

$$\Theta_{\pi_u(\varphi,\rho_\lambda)}(z\gamma) = \epsilon(G^u, A_G)\theta(z) \sum_Q R(G_{\gamma_s}^u, S_Q, 1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_* \theta_\lambda(\gamma_s)$$

where $S_Q = \mathrm{Cent}(Q, G^u)$ and the sum runs over any set of representatives for the $G_{\gamma_s}^u(F)$-conjugacy classes inside the $G^u(F)$-conjugacy class of Q_λ.

Proof: By [DR09, Lemmas 9.3.1,9.6.2] we know

$$\Theta_{\pi_u(\varphi,\rho_\lambda)}(z\gamma) = \epsilon(G^u, S_\lambda)\theta(z)R(G, S_\lambda, \theta_\lambda)(\gamma)$$

We will apply [DR09, Lemma 10.0.4] to the last factor, but first we want to study the indexing set of the sum appearing in the formula of that lemma. This indexing set is

$$Y := \{(S', \theta') \in \mathrm{Ad}(G^u(F))(S_\lambda, \theta_\lambda) |\ \gamma_s \in S'\}/\mathrm{Ad}(G^u_{\gamma_s}(F))$$

First we claim that the map

$$\begin{array}{rcl} \mathrm{Ad}(G^u(F))Q_\lambda & \to & \mathrm{Ad}(G^u(F))(S_\lambda, \theta_\lambda) \\ Q & \mapsto & \varphi_{Q_\lambda, Q}(S_\lambda, \theta_\lambda) \end{array}$$

is a bijection. It is clearly well-defined, and is moreover surjective because if $\mathrm{Ad}(g)(S_\lambda, \theta_\lambda)$ belongs to the right hand side, then $\mathrm{Ad}(g)Q_\lambda$ belongs to the left hand side and is a preimage. For the injectivity let

$$(S', \theta') = \varphi_{Q_\lambda, Q}(S_\lambda, \theta_\lambda) = \varphi_{Q_\lambda, Q'}(S_\lambda, \theta_\lambda)$$

Then $\varphi_{Q',Q} \in \Omega(S', G^u)$ and $\varphi_{Q',Q}\theta' = \theta'$. Since θ' is regular, $\varphi_{Q',Q} = 1$ and thus $Q' = Q$.

This proves the claimed bijectivity. Moreover, since $\varphi_{Q_\lambda,Q}(S_\lambda) = S_Q$ and

$$\gamma_s \in S_Q \Rightarrow S_Q \subset G_{\gamma_s} \Rightarrow Q \in \mathrm{Lie}(S_Q) \subset \mathrm{Lie}(G^u_{\gamma_s}) \Rightarrow \gamma_s \in S_Q$$

we see that our bijection restricts to the bijection

$$\begin{array}{rcl} \mathrm{Ad}(G^u(F))Q_\lambda \cap \mathrm{Lie}(G^u_{\gamma_s})(F) & \to & \{(S', \theta') \in \mathrm{Ad}(G^u(F))(S_\lambda, \theta_\lambda) |\ \gamma_s \in S'\} \\ Q & \mapsto & \varphi_{Q_\lambda, Q}(S_\lambda, \theta_\lambda) \end{array}$$

Both sides of this bijection carry a natural action of $G^u_{\gamma_s}(F)$ and that the bijection is equivariant with respect to these actions. Thus if we put

$$Y' := [\mathrm{Ad}(G^u(F))Q_\lambda \cap \mathrm{Lie}(G^u_{\gamma_s})(F)]/\mathrm{Ad}(G^u_{\gamma_s}(F))$$

we obtain a bijection

$$Y' \to Y$$

Applying now [DR09, Lemma 10.0.4] we obtain

$$\Theta_{\rho_\lambda}(z\gamma) = \epsilon(G^u, S_\lambda)\theta(z) \sum_{[Q]\in Y'} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_*\theta_\lambda(\gamma_s)$$

To complete the lemma, we only need to observe that since S_λ/Z is anisotropic, the maximal split subtorus of S_λ is A_G and thus $\epsilon(G^u, S_\lambda) = \epsilon(G^u, A_G)$. $\square$

We are now ready to establish the reduction formula for the t-unstable character.

Proposition 5.3.2. *Let $Q_0 \in \mathrm{Lie}(S_0)(F)$ be a regular semi-simple element, $\theta_0 = \mathrm{Ad}(q_0)_*\theta$, and t_{q_0} be the image of t under the dual of $\mathrm{Ad}(q_0^{-1})$. Then for any $\gamma \in G^u_{\mathrm{sr}}(F)_0$ and any $z \in Z(F)$ we have*

$$\Theta^t_{\rho,u}(z\gamma) = \epsilon(G, A_G)\theta(z) \quad \cdot \quad \sum_P [\varphi_{Q_0,P}]_*\theta_0(\gamma_s) \sum_Q \langle \mathrm{inv}(Q_0, Q), t_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u)$$

where P runs over a set of representatives for the $G^u_{\gamma_s}$-stable classes of elements of $\mathrm{Lie}(G^u_{\gamma_s})(F)$ which are G^u-stable conjugate to Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

Proof: For each $\lambda \in r^{-1}(u)$ put $\theta_\lambda = \mathrm{Ad}(q_\lambda)_*\theta$ and $Q_\lambda = \mathrm{Ad}(q_\lambda q_0^{-1})Q_0$. Then by Lemma 5.2.2 we know that $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$ is regular semi-simple, and so applying Lemma 5.3.1 and using the transitivity of the sign $\epsilon(\cdot,\cdot)$ we obtain

$$\Theta^t_{\rho,u} = \epsilon(G,A_G)\theta(z)\sum_{\lambda\in[r^{-1}(u)]}\lambda(t)\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_*\theta_\lambda(\gamma_s)$$

where the sum runs over the $G^u_{\gamma_s}(F)$-conjugacy classes inside the intersection of the $G^u(F)$-conjugacy class of Q_λ with $\mathrm{Lie}(G^u_{\gamma_s})(F)$. We obviously have

$$[\varphi_{Q_\lambda,Q}]_*\theta_\lambda = [\varphi_{Q_\lambda,Q}]_*[\varphi_{Q_0,Q_\lambda}]_*\theta_0 = [\varphi_{Q_0,Q}]_*\theta_0$$

and thus

$$\Theta^t_{\rho,u} = \epsilon(G,A_G)\theta(z)\sum_{\lambda\in[r^{-1}(u)]}\lambda(t)\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_0,Q}]_*\theta_0(\gamma_s)$$

Applying again Lemma 5.2.2 we obtain

$$\Theta^t_{\rho,u} = \epsilon(G,A_G)\theta(z)\sum_{Q'}\langle\mathrm{inv}(Q_0,Q'),t_{q_0}\rangle^{-1}\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_0,Q}]_*\theta_0(\gamma_s)$$

where now Q' runs over a set of representatives for the $G^u(F)$-classes inside the G^u-stable class of Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-classes inside the intersection of the $G^u(F)$-class of Q' with $\mathrm{Lie}(G^u_{\gamma_s})(F)$.

For any Q' in the first summation set and Q in the second, we have

$$\mathrm{inv}(Q_0,Q') = \mathrm{inv}(Q_0,Q)$$

because Q' and Q are $G^u(F)$-conjugate. Thus we obtain

$$\Theta^t_{\rho,u} = \epsilon(G,A_G)\theta(z)\sum_Q\langle\mathrm{inv}(Q_0,Q),t_{q_0}\rangle^{-1}R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_0,Q}]_*\theta_0(\gamma_s)$$

where now Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-classes inside the intersection of the stable class of Q_0 with $\mathrm{Lie}(G^u_{\gamma_s})(F)$.

Now consider two elements Q_1, Q_2 in the above summation set, and assume that they are $G^u_{\gamma_s}$-conjugate. This means that $\varphi_{Q_1,Q_2} = \mathrm{Ad}(g)$ with $g \in G^u_{\gamma_s}$. Since $\gamma_s \in S_{Q_1}$ the expression $\varphi_{Q_1,Q_2}(\gamma_s)$ is defined and we conclude that it equals γ_s. Thus

$$\begin{aligned}
[\varphi_{Q_0,Q_1}]_*\theta_0(\gamma_s) &= [\varphi_{Q_2,Q_1}]_*[\varphi_{Q_0,Q_2}]_*\theta_0(\gamma_s)\\
&= [\varphi_{Q_0,Q_2}]_*\theta_0(\varphi^{-1}_{Q_2,Q_1}(\gamma_s))\\
&= [\varphi_{Q_0,Q_2}]_*\theta_0(\gamma_s)
\end{aligned}$$

Rearranging terms again we arrive at the desired formula for $\Theta^t_{\rho,u}$. □

6 Character identities

In this section we assume all the notation established in the previous sections, in particular all parts of Section 3. Our goal is to prove Theorem 3.4.2.

6.1 Beginning of the proof of 3.4.2

Lemma 6.1.1. *Let D be a diagonalizable group defined over F and split over F^u. Then*

$$D(F) = D_s(F) \cdot \left[D(F) \cap \bigcap_{\chi \in X^*(D)} [\ker(v \circ \chi)] \right]$$

where D_s is the maximal split subtorus of D.

Proof: For any $x \in D(F^u)$ the map

$$\lambda_x : X^*(D) \to \mathbb{Z}, \qquad \chi \mapsto v(\chi(x))$$

is $\mathbb{Z}$-linear. Sending x to λ_x defines a homomorphism

$$D(F^u) \to X_*(D)$$

which is Γ-equivariant because of the Γ-invariance of $v : [F^u]^\times \to \mathbb{Z}$. A right inverse of this homomorphism is given by evaluation at π.

Now let $x \in D(F)$. Then $\lambda_x \in X_*(D)$ is Γ-fixed, and so its image $y = \pi^{\lambda_x} \in D(F^u)$ under the right inverse lies in $D_s(F)$. Thus $x = y \cdot (xy^{-1})$ is the desired decomposition. $\square$

Lemma 6.1.2. *Assume that γ does not belong to $Z(F)G^u(F)_0$. Then both sides of* (3.2) *vanish.*

Proof: The left hand vanishes by [DR09, Lemma 9.3.1]. Turning to the right hand side, assume by way of contradiction that some γ^H in the summation set of (3.2) lies in $A_H(F)H(F)_0$, and write $\gamma^H = zx$. The admissible isomorphism $\varphi_{\gamma^H,\gamma}$ sends x into $G^u(F)_0$ and because H is elliptic it maps $A_H(F)$ to $A_G(F)$. Thus $\gamma = \varphi_{\gamma^H,\gamma}(\gamma_H) \in A_G(F)G^u(F)_0$ contradicting the assumption of the lemma. We conclude that all γ^H occurring in the summation set of (3.2) lie outside of $A_H(F)H(F)_0$. But by the previous lemma, $Z_H(F)H(F)_0 = A_H(F)H(F)_0$, because the set

$$Z_H(F) \cap \bigcap_{\chi \in X^*(Z_H)} [\ker(v_F \circ \chi)]$$

lies in the maximal bounded subgroup of $T_0^H(F)$. By [DR09, Lemma 9.3.1] the left hand side of (3.2) also vanishes. $\square$

Lemma 6.1.3. *The isomorphism*

$$T_0^w \xrightarrow{\eta} T_0^{H,w^H}$$

is defined over F. If $\gamma \in T_0^w(F)$ is topologically semi-simple and $z \in Z^\circ(F)$ then

$$\theta(\gamma) = \theta^H(\eta(\gamma)) \qquad \theta(z) = \theta^H(\eta(z))\lambda^G(z)$$

where $\lambda^G : Z^\circ(F) \to \mathbb{C}^\times$ is the character of [LS87, Lemma 4.4.A].

Proof: Recall that T_0^w is the torus whose complex dual is given by the complex torus $\widehat{T}_0$ with Γ-action

$$\sigma(t) = \mathrm{Ad}(\varphi(\sigma))t$$

for all $\sigma \in W_F, t \in \widehat{T}_0(\mathbb{C})$ where the conjugation takes place in ${}^LG$. Analogously we have the torus T_0^{H,w^H} whose complex dual is the complex torus $\widehat{T}_0^H$ with Γ-action

$$\sigma(t) = \mathrm{Ad}(\varphi^H(\sigma))t$$

for all $\sigma \in W_F, t \in \widehat{T}_0^H(\mathbb{C})$ where now the conjugation takes place in ${}^LH$. The statement that

$$\eta : T_0^w \to T_0^{H,w^H}$$

is defined over F is equivalent to the statement that the isomorphism of complex tori

$$\widehat{\eta} : \widehat{T}_0^H \to \widehat{T}_0$$

is equivariant with respect to the above actions. But

$$\begin{aligned}
\widehat{\eta}(\mathrm{Ad}(\varphi^H(\sigma))t) &= {}^L\eta(\mathrm{Ad}(\varphi^H(\sigma))t) \\
&= \mathrm{Ad}({}^L\eta\varphi^H(\sigma)){}^L\eta(t) \\
&= \mathrm{Ad}(\varphi(\sigma))\widehat{\eta}(t)
\end{aligned}$$

This proves the first assertion.

The restriction of θ to the maximal bounded subgroup of T_0^ω, to which γ belongs, is determined by the restriction of the Langlands parameter φ_T of θ to inertia, which by construction is simply given by the restriction to inertia of $\varphi = {}^L\eta \circ \varphi^H$. This restriction is the cocycle

$$I_F \xrightarrow{\varphi^H} {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow \widehat{G}$$

which by construction lands in $\widehat{T}_0$. Since ${}^L\eta$ is trivial on inertia, we see that this is the same as the cocycle

$$I_F \xrightarrow{\varphi^H} {}^LH \longrightarrow \widehat{H} \xrightarrow{\widehat{\eta}} \widehat{G}$$

which also lands in $\widehat{T}_0$ and equals the restriction to inertia of $\widehat{\eta} \circ \varphi_{T^H}$. The latter is the cocycle determining the restriction of $\theta^H \circ \eta$ to the maximal bounded subgroup of T_0^ω. This proves the second assertion.

Let T be any torus of G coming from H. In [LS87, §3.5] Langlands and Shelstad construct an element $a \in H^1(W_F, \widehat{T})$. The character $\lambda^G(z)$ is then the restriction to $Z^\circ(F)$ of the character on $T(F)$ corresponding via the Langlands correspondence to a. The construction of a involves χ-data, but one sees easily that its image under

$$H^1(W_F, \widehat{T}) \to H^1(W_F, \widehat{Z^\circ})$$

is independent of that choice and is in fact represented by the cocycle

$$W_F \hookrightarrow {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow {}^L[Z^\circ] \longrightarrow \widehat{Z^\circ}$$

By construction of the Langlands parameter φ_T of θ, the restriction of θ to $Z^\circ(F)$ is given by the cocycle

$$W_F \xrightarrow{\varphi^H} {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow {}^L[Z^\circ] \longrightarrow \widehat{Z^\circ}$$

while that of $\theta^H \circ \eta$ is given by the cocycle

$$W_F \xrightarrow{\varphi^H} {}^L H \longrightarrow \widehat{H} \xrightarrow{\widehat{\eta}} \widehat{G} \longrightarrow \widehat{Z^\circ}$$

It is clear that of these three cocycles, the second one equals the product of the first and the third, which implies the final statement of the lemma. □

Corollary 6.1.4. *If equation* (3.2) *holds for all strongly regular semi-simple* $\gamma \in G^u(F)_0$*, then it holds for all strongly-regular semi-simple* $\gamma \in G^u(F)$*.*

Proof: By Lemma 6.1.2 equation (3.2) holds trivially if γ does not belong to $Z(F)G^u(F)_0$. By Lemma 6.1.1 we have $Z(F)G^u(F)_0 = A_G(F)G^u(F)_0$, so it is enough to prove equation (3.2) for strongly regular semi-simple elements $\gamma = z\gamma'$ with $z \in A_G(F)$ and $\gamma' \in G^u(F)_0$. By Proposition 5.3.2 we know the behavior of the unstable character under central translations, namely $\Theta^s_{\rho,u}(z\gamma) = \theta(z)\Theta^s_{\rho,u}(\gamma)$ and thus using our assumption we have

$$\begin{aligned}
\Theta^s_{\rho,u}(z\gamma) &= \theta(z) \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H) \\
&= \theta^H(\eta(z))\lambda^G(z) \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H)
\end{aligned}$$

where for the second equality we have invoked Lemma 6.1.3. Using [LS87, Lemma 4.4.A] and the obvious invariance of the terms $D(\gamma)$ and $D^H(\gamma^H)$ under central translations this can be written as

$$\begin{aligned}
&= \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\eta(z)\gamma^H, z\gamma) \frac{D(\eta(z)\gamma^H)^2}{D(z\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\eta(z)\gamma^H) \\
&= \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, z\gamma) \frac{D(\gamma^H)^2}{D(z\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H)
\end{aligned}$$

which was to be shown. □

6.2 A reduction formula for the endoscopic lift of the stable character

Lemma 6.2.1. *Let J be an unramified F-group and $y \in J(F)$ be a topologically semi-simple element belonging to a hyperspecial maximal compact subgroup. Let γ be an element of either $J(F)$ or $\mathrm{Lie}(J)(F)$ for which $\mathrm{Cent}(\gamma, J) \subset J_y$. Then the finite group $\pi_0(J^y(F))$ acts simply transitively on the set of J_y-stable classes inside the J^y-stable class of γ.*

Proof: Clearly $J^y(F)$ acts on the J^y-stable class of γ, and $J_y(F)$ preserves each J_y-stable class inside, so that we obtain an action of $\pi_0(J^y)(F)$ on the set of J_y-stable classes inside the J^y-stable class of γ.

Consider the sequence

$$1 \to J_y(F) \to J^y(F) \to \pi_0(J^y)(F) \to H^1(F, J_y) \to H^1(F, J^y)$$

By [Kot86, Prop 7.1] the last arrow has trivial kernel, which implies that the third arrow is surjective, so that we have a short exact sequence

$$1 \to J_y(F) \to J^y(F) \to \pi_0(J^y)(F) \to 1$$

Let γ' be J^y-stably conjugate to γ, and pick $g \in J^y(\overline{F})$ s.t. $\mathrm{Ad}(g)\gamma = \gamma'$ and $g^{-1}\sigma(g) \in J_\gamma \subset J_y$ for any $\sigma \in \Gamma$. Then the image $\bar{g} \in \pi_0(J^y)$ of g belongs to $\pi_0(J^y)(F)$. Let $h \in J^y(F)$ be a preimage of $\bar{g}$. Then $\mathrm{Ad}(h)\gamma$ and γ' are stably conjugate by $gh^{-1} \in J_y(\overline{F})$. This proves transitivity.

To show simplicity, let γ' by J_y-stably conjugate to γ and pick $h \in J_y(\overline{F})$ s.t. $\mathrm{Ad}(h)\gamma = \gamma'$. If $g \in J^y(F)$ is also an element s.t. $\gamma' = \mathrm{Ad}(g)\gamma$, then $gh^{-1} \in \mathrm{Cent}(\gamma, J) \subset J_y$ so $g \in J_y(\overline{F}) \cap J^y(F) = J_y(F)$. □

Remark: The same proof shows that under the same assumptions, $\pi_0(J^y(F))$ acts simply transitively on the set of $\mathrm{Ad}J_y(\overline{F})$-orbits in $\mathrm{Ad}J^y(\overline{F})\gamma \cap J_y(F)$.

Lemma 6.2.2. *Let $\gamma' \in G^u(F)$ be a strongly-regular semi-simple element. Assume that for some $\lambda \in r^{-1}(u)$ we have $\gamma'_s \in S_\lambda(F)$. Then there exists a $\gamma \in G(F)$ stably conjugate to γ' s.t. $\gamma_s \in S_0(F)$.*

Proof: By construction we know that $\mathrm{Ad}(q_0 q_\lambda^{-1}) : S_\lambda \to S_0$ is an isomorphism over F. Put $t = \mathrm{Ad}(q_0 q_\lambda^{-1})\gamma'_s$. Then t, being topologically semi-simple, belongs to the maximal bounded subgroup of $S_0(F)$ and thus lies in $G(O_F)$. It follows form [Kot86, Prop. 7.1] that G_t is quasi-split. The map $\mathrm{Ad}(q_0 q_\lambda^{-1}) : G^u_{\gamma_s} \to G_t$ is a twist and so there exists an $i \in G_t(\overline{F})$ s.t. if $T' = \mathrm{Cent}(\gamma', G^u_{\gamma'_s})$ and $f = \mathrm{Ad}(i q_0 q_\lambda^{-1})$ then the torus $T := f(T')$ and the isomorphism $f : T' \to T$ are defined over F. Put $\gamma = f(\gamma')$. By construction $\gamma_s = t$ and f is a (ψ, u)-equivalent twist, so γ is the element we want. □

Remark: The same proof can be applied to an element $\gamma^H \in H(F)$ and the trivial twist $(\mathrm{id}, 1) : H \to H$.

Lemma 6.2.3. *Let $\gamma \in G(F)_0$ and $\gamma^H \in H(F)_0$ be a pair of related strongly G-regular elements s.t. $\gamma_s \in S_0(F)$ and $\gamma^H_s \in S^H_0(F)$. Then the admissible isomorphism $\varphi_{\gamma^H,\gamma}$ makes $H_{\gamma^H_s}$ into an endoscopic group for G_{γ_s}. If Δ_0 and Δ'_0 denote the transfer factors for (G, H) and $(G_{\gamma_s}, H_{\gamma^H_s}, \varphi_{\gamma^H,\gamma})$ normalized with respect to admissible splittings (in the sense of [Hal93, §7]) then one has*

$$\Delta_0(\gamma^H, \gamma) = \Delta'_0(\gamma^H_u, \gamma_u)$$

Proof: By [Kot86, Prop. 7.1] both groups $H_{\gamma^H_s}$ and G_{γ_s} are unramified, so they fall in the framework of [Hal93] and one has the normalization Δ'_0 of the transfer factor with respect to an admissible splitting. We want to apply [Hal93, Thm. 10.18] to conclude

$$\Delta_0(\gamma^H, \gamma) = \Delta'_0(\gamma^H, \gamma)$$

This theorem has two requirements. One is $p > e_G + 1$, which is given in the statement of the theorem, and which we are assuming. The other one is $\gamma \in G(O_F)$ and $\gamma^H \in H(O_F)$, which is a general requirement for the whole section 10 in loc. cit. However, tracing through the arguments of that section one sees that up to the proof of Thm. 10.18, the only property of γ and γ^H which is used is that fact that they are compact and so have a topological Jordan decomposition, together with the fact that homomorphisms preserve the

topological Jordan decomposition and the knowledge of its explicit form for elements of extensions of F. In the proof of Thm. 10.18 the elements γ^H and γ are replaced by high powers of themselves, let's call them γ'^H and γ', which are very close (and can be made arbitrarily close) to γ_s^H and γ_s. Then Lemma 8.1. of loc. cit. is involved for the pair (γ'^H, γ'). For that Lemma it is essential that $\gamma'^H \in H(O_F)$ and $\gamma' \in G(O_F)$. But from our assumptions it follows that $\gamma_s^H \in H(O_F)$ and $\gamma_s \in G(O_F)$, and since these groups are open, and the elements γ'^H and γ' can be made arbitrarily close to γ_s^H and γ_s, Lemma 8.1 can be invoked.

Thus we conclude that

$$\Delta_0(\gamma^H, \gamma) = \Delta_0'(\gamma^H, \gamma)$$

By [LS90, §3.5] there exists a character $\lambda : Z_{G_{\gamma_s}}(F) \to \mathbb{C}^\times$ s.t. for all strongly regular elements $z \in H_{\gamma_s^H}(F)$ and $w \in G_{\gamma_s}(F)$ one has

$$\Delta_0'(z\gamma_s^H, w\gamma_s) = \lambda(\gamma_s)\Delta_0'(z, w)$$

Thus

$$\frac{\Delta_0'(\gamma^H, \gamma)}{\Delta_0'(\gamma_u^H, \gamma_u)} = \lambda(\gamma_s) = \frac{\Delta_0'(z\gamma_s^H, w\gamma_s)}{\Delta_0'(z, w)}$$

We choose w to lie in an unramified torus $T \subset G_{\gamma_s}$. Then

$$\frac{\Delta_0'(z\gamma_s^H, w\gamma_s)}{\Delta_0'(z, w)} = \langle a, \gamma_s \rangle$$

where $\langle a, \cdot \rangle$ is a character $T(F) \to \mathbb{C}^\times$ constructed in [LS87, §3.5]. By [Hal93, Lemma 11.2] this character is unramified, and thus takes trivial value at γ_s. It follows that

$$\Delta_0'(\gamma^H, \gamma) = \Delta_0'(\gamma_u^H, \gamma_u)$$

and the proof is complete. □

Lemma 6.2.4. *For $\gamma \in G(F)_0$ the expression*

$$\sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_0(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H) \tag{6.1}$$

is equal to

$$\sum_y \sum_\xi |\pi_0(H^y(F))|^{-1} \sum_{z \in H_y(F)_{\mathrm{sr}}/\mathrm{st}} \Delta_{0,y,\xi}(z, \gamma_u) \frac{D_{H_y}(z)^2}{D_{G_{\gamma_s}}(\gamma_u)^2} \mathcal{S}\Theta_{\varphi^H}(yz) \tag{6.2}$$

where y runs over a subset of $S_0^H(F)$ consisting of representatives for the stable classes of preimages of γ_s which lie in $S_0^H(F)$, ξ runs over the (H_y, G_{γ_s})-equivalence classes of admissible embeddings mapping y to γ_s, and $\Delta_{0,y,\xi}$ is the absolute transfer factor for (H_y, G_{γ_s}, ξ) normalized with respect to an admissible splitting.

Proof: The sum of the first expression runs over the set of stable classes of strongly-regular semi-simple elements in $H(F)$. If $\gamma^H \in H(F)$ is strongly-regular semi-simple, but γ_s^H does not lie in a torus which is stably conjugate to S_0^H, then by Proposition 5.3.2 we have $\mathcal{S}\Theta_{\varphi^H}(\gamma^H) = 0$. Moreover if γ_s^H is not a preimage of γ_s, then γ^H is not a preimage of γ and so $\Delta_0(\gamma^H, \gamma) = 0$. Thus if $\Gamma^H \subset H(F)$ is the set of strongly-regular semi-simple elements γ^H for which

γ_s^H is a preimage of γ_s and lies in a torus stably conjugate to S_0^H, then we may restrict the summation in the first expression to Γ^H/st. Let $Y \subset S_0^H(F)$ be a set of representatives for the stable classes of those elements of $S_0^H(F)$ which occur as the topologically semi-simple parts of elements in Γ^H. We claim that we have a surjective map

$$p : \Gamma^H/\mathrm{st} \to Y$$

By Lemma 6.2.2 and the remark thereafter every stable class $\mathcal{C} \subset \Gamma^H$ has a representative γ^H s.t. $\gamma_s^H \in S_0^H(F)$. By construction there exists $y \in Y$ stably conjugate to γ_s^H. By [Kot86, Prop. 7.1] there exists $h \in H(O_F)$ s.t. $\mathrm{Ad}(h)\gamma_s^H = y$. But then $\mathrm{Ad}(h)\gamma^H \in \mathcal{C}$. We see that there are elements in $\mathcal{C}$ whose topologically semi-simple parts lie in Y. If $\gamma^H, \gamma'^H \in \mathcal{C}$ are two such elements, then their stable conjugacy implies the stable conjugacy of their topologically semi-simple parts, but by construction of Y this means that their topologically semi-simple parts are actually equal. Thus we may define $p(\mathcal{C})$ by choosing any $\gamma^H \in \mathcal{C}$ with $\gamma_s^H \in Y$ and sending it to γ_s^H.

Next we claim that for every $y \in Y$ we have a surjective map

$$[H_y(F)]_{(H,y)-\mathrm{sr,tu}}/\mathrm{st} \to p^{-1}(y), \qquad z \mapsto yz$$

whose fibers are torsors under $\pi_0(H^y(F))$. Here $[H_y(F)]_{(H,y)-\mathrm{sr,tu}}$ denotes the set of topologically unipotent elements $z \in H_y(F)$ for which yz is H-strongly regular, and st is stable conjugacy in H_y. It is immediate that this map is well-defined and surjective. We claim that each fiber constitutes a single H^y-stable class. If z, z' lie in the same fiber, then there exists $h \in H(\overline{F})$ s.t. $\mathrm{Ad}h(yz) = yz'$. But then $\mathrm{Ad}h(y) = y$, so $h \in H^y(\overline{F})$, and $\mathrm{Ad}h(z) = z'$, which shows that z, z' lie in the same H^y-stable class. Conversely if z, z' lie in the same H^y-stable class then they clearly lie in the same fiber. From Lemma 6.2.1 it now follows that the fibers are torsors under $\pi_0(H^y(F))$.

We conclude that expression (6.1) is equal to

$$\sum_{y\in Y} |\pi_0(H^y(F))|^{-1} \sum_{z\in[H_y(F)]_{(H,y)-\mathrm{sr,tu}}/\mathrm{st}} \Delta_0(yz,\gamma)\frac{D(yz)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\varphi^H}(yz) \tag{6.3}$$

Consider y, z contributing to the above expression. If (yz, γ) is not a pair of (G, H)-related elements, then $\Delta_0(yz, \gamma) = 0$. Now assume that (yz, γ) is a related pair. Then (z, γ_u) is a pair of $(G_{\gamma_s}, H_y, \varphi_{yz,\gamma})$-related elements, and from Lemma 6.2.3 we know that

$$\Delta_0(yz,\gamma) = \Delta_{0,y,\varphi_{yz,\gamma}}(z,\gamma_u)$$

Moreover, if ξ is a (G, H)-admissible embedding carrying y to γ_s but not equivalent to $\varphi_{yz,\gamma}$, the pair (z, γ_u) is not (G_{γ_s}, H_y, ξ)-related, and thus $\Delta_{0,y,\xi}(z, \gamma_u) = 0$. It follows that

$$\Delta_0(yz,\gamma) = \sum_{\xi} \Delta_{0,y,\xi}(z,\gamma_u)$$

where ξ runs over the set of (G_{γ_s}, H_y)-equivalence classes of (G, H)-admissible embeddings carrying y to γ_s. As in the proof of [Hal93, Lem. 8.1] we have

$$D(\gamma) = D_{G_{\gamma_s}}(\gamma_u) \qquad \text{and} \qquad D(yz) = D_{H_y}(z)$$

Thus expression (6.3) equals

$$\sum_{y\in Y}\sum_{\xi} |\pi_0(H^y(F))|^{-1} \sum_{z\in[H_y(F)]_{(H,y)-\mathrm{sr,tu}}/\mathrm{st}} \Delta_{0,y,\xi}(z,\gamma_u)\frac{D_{H_y}(z)^2}{D_{G_{\gamma_s}}(\gamma_u)^2}\mathcal{S}\Theta_{\varphi^H}(yz)$$

Finally, note that every $z \in H_y(F)_{\mathrm{sr}}$ which is a (G_{γ_s}, H_y, ξ)-preimage of γ_u automatically belongs to the set $[H_y(F)]_{(H,y)-\mathrm{sr},\mathrm{tu}}$. Hence we may extend the summation over z to all of $H_y(F)_{\mathrm{sr}}$. Also if $y \in S_0^H(F)$ is a preimage of γ_s but does not belong to Y, then $H_y(F)$ does not contain a (G_{γ_s}, H_y, ξ)-preimage of γ_u for any ξ, and thus the terms $\Delta_{0,y,\xi}(z, \gamma_u)$ vanish for all ξ and z. Hence we may add to Y representatives for the stable classes of such elements without changing the value of the sum. But then the expression we obtain is (6.2). □

Corollary 6.2.5. *If $\gamma \in G^u(F)_0$ is a strongly regular semi-simple element which does not have a stable conjugate $\gamma' \in G(F)_0$ with $\gamma'_s \in S_0(F)$, then both sides of Equation* (3.2) *vanish.*

Proof: Consider first the left hand side. In view of Proposition 5.3.2, it vanishes unless γ_s lies in the centralizer of some $Q \in \mathrm{Lie}(G^u)(F)$ stably conjugate to Q_0. But such a Q is then rationally conjugate to Q_λ for some $\lambda \in r^{-1}(u)$ and hence replacing γ by a rational conjugate we may assume $\gamma_s \in S_\lambda$. Thus, by Lemma 6.2.2, the non-vanishing of the left hand side of Equation (3.2) implies the existence of γ' as claimed.

We now turn to the right hand side. Let $\tilde{\gamma} \in G(F)_0$ be any stable conjugate of γ. By Lemma 6.2.4, the right hand side of Equation (3.2) vanishes at $\tilde{\gamma}$ unless there exists a triple (y, ξ, z) s.t. $y \in S_0^H(F)$ is a preimage of $\tilde{\gamma}_s$, ξ is a (G, H)-admissible embedding s.t. $\xi(y) = \tilde{\gamma}_s$, and $z \in H_y(F)$ is a $(G_{\tilde{\gamma}_s}, H_y, \xi)$-preimage of $\tilde{\gamma}_u$. By Lemma 6.1.3 the map

$$S_0^H \xrightarrow{\mathrm{Ad}(q_0^H)^{-1}} T_0^{H,w^H} \xrightarrow{\eta^{-1}} T_0^w \xrightarrow{\mathrm{Ad}(q_0)} S_0$$

is an admissible isomorphism defined over F. Let y' be the image of y under this isomorphism. Then $G_{y'}$ is quasi-split by [Kot86, Prop. 7.1] and so there exists $z' \in G_{y'}(F)$ which is an image of z. But then $\gamma' = y'z'$ is a stable conjugate of $\tilde{\gamma}$, hence of γ. Thus the non-vanishing of the right hand side of Equation (3.2) at $\tilde{\gamma}$ implies the existence of γ' as claimed. But since γ and $\tilde{\gamma}$ are stably conjugate, the non-vanishing of said expression at γ is equivalent to its non-vanishing at $\tilde{\gamma}$, since the value at $\tilde{\gamma}$ differs from the value at γ by a non-zero multiplicative factor. □

6.3 Lemmas about transfer factors

In this section G' is an unramified F-group and $(H', s, {}^L\eta)$ is an unramified extended endoscopic triple for G'. Let (T'_0, B'_0) be a Borel pair of G' over F. We choose hyperspecial points in the buildings of G' and H', s.t. the one for G' lies in the apartment of T'_0. We also choose an admissible splitting $(T'_0, B'_0, \{X'_\alpha\})$ for G' in the sense of [Hal93, §7]. Then we have the transfer factors normalized with respect to that splitting both on the group level ([LS87, §3.7]), as well as on the Lie algebra level ([Kot99]). We will call both these transfer factors Δ_0, as there will be no possibility of confusion between the two.

Lemma 6.3.1. *For any semi-simple strongly regular topologically unipotent $\gamma^H \in H'(F)$ and $\gamma \in G'(F)$, we have*

$$\Delta_0(\gamma^H, \gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2} = \Delta_0(\log(\gamma^H), \log(\gamma))\frac{D(\log(\gamma^H))}{D(\log(\gamma))}$$

Proof: We choose a positive integer m with the property that the sequences

$$\gamma_k = \gamma^{p^{km}} \qquad \gamma_k^H = [\gamma^H]^{p^{km}}$$

converge to 1 (cf. [DR09, §7]), and put

$$X^H = \log(\gamma^H), X = \log(\gamma), X_k^H = \log(\gamma_k^H), X_k = \log(\gamma_k)$$

As argued in [Hal93, §10] we have

$$\Delta_0(\gamma_{2k}^H, \gamma_{2k}) \frac{D(\gamma_{2k}^H)^2}{D(\gamma_{2k})^2} = |p^{km}|^{-N} \Delta_0(\gamma^H, \gamma)$$

where N is the number of roots in G' outside H' and $|\ |$ is the unique absolute value on $\overline{F}$ extending that on F. By the same arguments one also has

$$\Delta_0(X_{2k}^H, X_{2k}) \frac{D(X_{2k}^H)}{D(X_{2k})} = |p^{km}|^{-N} \Delta_0(X^H, X) \frac{D(X^H)}{D(X)}$$

Thus it will be enough to show the equality claimed in the lemma with γ^H, γ replaced by $\gamma_{2k}^H, \gamma_{2k}$ for some k which we may freely choose.

As argued in [Wal97, §2.3], there exists a positive integer K s.t. for all $k > K$

$$\Delta_0(\gamma_{2k}^H, \gamma_{2k}) \frac{D(\gamma_{2k}^H)}{D(\gamma_{2k})} = \Delta_0(X_{2k}^H, X_{2k})$$

We now claim that, after potentially increasing K, we have

$$D(\gamma_{2k}^H) = D(X_{2k}^H) \qquad D(\gamma_{2k}) = D(X_{2k})$$

For this it is enough to show that if $T \subset G'$ is a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}'$ and $Y \in \mathfrak{t}(F)$ is small enough then for all roots $\alpha \in R(T, G')$ we have

$$|\alpha(\exp(Y)) - 1| = |d\alpha(Y)|$$

Let E/F be the extension splitting T, and let v_E be the unique valuation E extending that on F (here we deviate from our usual notation). Then

$$|\alpha(\exp(Y)) - 1| = |\exp(d\alpha(Y)) - 1| = |d\alpha(Y) + \sum_{k>1} \frac{d\alpha(Y)^k}{k!}|$$

Putting $u = d\alpha(Y)$, we have by a computation similar to the proof of [DR09, B.1.1]

$$v_E(\frac{u^k}{k!}) = kv_E(u) - eA(k) > kv_E(u) - (k-1)$$

where as in loc. cit. $A(k) = \sum_{i>0} \lfloor \frac{k}{p^i} \rfloor$ and e is the ramification degree of $F/\mathbb{Q}_p$. Thus if $v_E(u) \geq 1$ then for all $k > 1$

$$v_E(\frac{u^k}{k!}) > v_E(u)$$

from which follows

$$|u + \sum_{k>1} \frac{u^k}{k!}| = |u|$$

This finishes the proof of the claim about D and the lemma follows. $\square$

Lemma 6.3.2. *Let $S \subset G'$ be a torus (as usual defined over F) which is defined and split over O_{F^u}. Let $Q \in \mathrm{Lie}(S)(O_{F^u}) \cap \mathfrak{g}'(F)$ be semi-simple regular, and Q^H be any preimage of Q in $\mathfrak{h}'(F)$. Then*

$$\Delta_0(Q^H, Q) = 1$$

Proof: For $\alpha \in R(S, G')$ let $a_\alpha = d\alpha(Q)$. As Kottwitz observes in [Kot99], this defines a-data for $R(S, G')$ and with respect to that a-data, $\Delta_{II}(Q^H, Q) = 1$. To show that $\Delta_I(Q^H, Q) = 1$ we adapt the argument of [Hal93, Lem. 7.2]. Since S is defined and split over O_{F^u}, all roots of S are defined over O_{F^u} and we have $a_\alpha \in O_{F^u}$. Then as in loc. cit. we see that the cocycle $m(\sigma_S)$ constructed in [LS87, §2.3] takes values in $G'(O_{F^u})$. Since the torus T_0' is also defined over O_{F^u}, there exists $g \in G'(O_{F^u})$ s.t. $S = \mathrm{Ad}(g)T_0'$. Thus the cocycle $\mathrm{Ad}(g)^{-1}m(\sigma_S)$ of Γ in $S(\overline{F})$ takes values in $S(O_{F^u})$ and is thus cohomologically trivial. But $\Delta_I(Q^H, Q)$ is the value of a character on $H^1(\Gamma, S)$ at that cocycle. □

Lemma 6.3.3. *Let $\gamma^H \in H'(F)$ and $\gamma \in G'(F)$ be semi-simple, strongly regular, and topologically unipotent. Then γ is an image of γ^H if and only if $\log(\gamma)$ is an image of $\log(\gamma^H)$.*

Proof: We define $\gamma_k^H, \gamma_k, X^H, X, X_k^H, X_k$ as in the proof of Lemma 6.3.1. It is clear that γ is an image of γ^H if and only if γ_k is an image of γ_k^H for some (then any) k. The same holds for the X's. This reduces the proof to the case where the elements are near the identity, in which case it is clear. □

6.4 Completion of the proof of theorem 3.4.2

By Corollaries 6.1.4 and 6.2.5 it is enough to prove Equation (3.2) for all strongly regular semi-simple elements $\gamma \in G^u(F)_0$ which have a stable conjugate $\gamma' \in G(F)_0$ s.t. $\gamma_s' \in S_0(F)$. Fix such a pair γ, γ' and consider the value at γ of the right hand side of Equation (3.2):

$$\sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\varphi^H}(\gamma^H) \tag{6.4}$$

By construction of $\Delta_{\psi,u}$ we have

$$\Delta_{\psi,u}(\gamma^H, \gamma) = \epsilon_L(V, \psi)\Delta_0(\gamma^H, \gamma')\langle \mathrm{inv}(\gamma', \gamma), \widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}$$

where Δ_0 is the absolute transfer factor for (G, H) normalized with respect to our chosen splitting. By Lemma 6.1.3 the map

$$S_0^H \xrightarrow{\mathrm{Ad}(q_0^H)^{-1}} T_0^{H,w^H} \xrightarrow{\eta^{-1}} T_0^w \xrightarrow{\mathrm{Ad}(q_0)} S_0$$

is an admissible isomorphism defined over F. We fix $Q_0 \in \mathrm{Lie}(S_0)(F)$ satisfying the requirements of the element X_S in [DR09, Lemma 12.4.3], and let Q_0^H be the preimage of Q_0 under this embedding. Then Q_0^H also satisfies the same requirements.

We now apply Lemma 6.2.4 and Proposition 5.3.2 to conclude that (6.4) equals

$$\epsilon_L(V,\psi)\epsilon(H,A_H)\sum_y\sum_\xi|\pi_0(H^y(F))|^{-1}\sum_{P^H}[\varphi_{Q_0^H,P^H}]_*\theta_0^H(y)$$
$$\langle\mathrm{inv}(\gamma',\gamma),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}\sum_{z\in H_y(F)_{\mathrm{sr}}/\mathrm{st}}\Delta_{0,y,\xi}(z,\gamma'_u)\frac{D_{H_y}(z)^2}{D_{G_{\gamma'_s}}(\gamma'_u)^2}$$
$$\sum_{Q^H}R(H_y,S_{Q^H},1)(z) \tag{6.5}$$

Let us recall the summation sets. y runs over a set $Y\subset S_0^H(F)$ representing the stable classes of preimages of γ'_s which intersect $S_0^H(F)$, ξ runs over the $(G_{\gamma'_s},H_y)$-equivalence classes of (G,H)-admissible embeddings which map y to γ'_s, P^H runs over a set of representatives for the H_y-stable classes of elements of $\mathrm{Lie}(H_y)(F)$ which are H-stably conjugate to Q_0^H, z runs over the stable classes of strongly regular elements in $H_y(F)$, and Q^H runs over a set of representatives for the $H_y(F)$-classes inside the H_y-stable class of P^H.

Consider a triple (y,ξ,P^H). Since $G_{\gamma'_s}$ is quasi-split, there exists an $(G_{\gamma'_s},H_y,\xi)$-image $P'\in\mathrm{Lie}(G_{\gamma'_s})(F)$ of P^H, unique up to stable conjugacy. We claim that the map
$$p:(y,\xi,P^H)\mapsto P'$$
is a surjection from the set of triples (y,ξ,P^H) occurring in (6.5) to the set of $G_{\gamma'_s}$-stable classes of elements of $\mathrm{Lie}(G_{\gamma'_s})(F)$ stably conjugate to Q_0, and moreover that the fiber of this surjection through (y,ξ,P^H) is a torsor under $\pi_0(H^y(F))$ for the action of this group by conjugation on all factors of the triple (the first factor is of course fixed by this action).

To see surjectivity, choose P' in the target of p. Let $\tilde y=\varphi_{P',Q_0^H}(\gamma'_s)$. There exists a $y\in Y$ stably conjugate to $\tilde y$. By [Kot86, Prop. 7.1] there exists $h\in H(O_F)$ s.t. $\mathrm{Ad}(h)\tilde y=y$. Put $P^H=\mathrm{Ad}(h)Q_0^H$. Then $(y,\varphi_{P^H,P'},P^H)$ is a preimage of P' under p.

Now let (y,ξ,P^H) be an element in the source of p and let P' be its image. We claim that the map
$$\tilde p:\tilde P^H\mapsto(y,\varphi_{\tilde P^H,P'},\tilde P^H)$$
is an $\pi_0(H^y(F))$-equivariant bijection from the set of H_y-stable classes inside the H^y-stable class of P^H to the fiber of p through (y,ξ,P^H). Once this has been shown, the claim about the fibers of p will follow from Lemma 6.2.1.

Indeed, let $\tilde P^H$ be H^y-stably conjugate to P^H. Then $\varphi_{\tilde P^H,P^H}(y)=y$ and moreover since P' is a $(G_{\gamma'_s},H_y,\xi)$-image of P^H we have $\varphi_{P^H,P'}(y)=\gamma'_s$. This implies $\varphi_{\tilde P^H,P'}(y)=\gamma'_s$ and we see that $(y,\varphi_{\tilde P^H,P'},\tilde P^H)$ belongs to the target of the proposed map $\tilde p$. If $\tilde P^H$ is replaced by an H_y-stable conjugate, then $\varphi_{\tilde P^H,P'}$ remains within its equivalence class. We see that $\tilde p$ is a well-defined and $\pi_0(H^y(F))$-equivariant map as claimed. It is clearly injective. To show surjectivity, let $(\tilde y,\tilde\xi,\tilde P^H)\in p^{-1}(P')$. By definition of the map p, we must have that $\tilde\xi$ and $\varphi_{\tilde P^H,P'}$ are $(G_{\gamma'_s},H_{\tilde y})$-equivalent and $\tilde y=\varphi_{P',\tilde P^H}(\gamma'_s)$ and so we only have to show that $\tilde P^H$ and P^H are H^y-stably conjugate. We have $\varphi_{P^H,P'}(y)=\gamma'_s=\varphi_{\tilde P^H,P'}(\tilde y)$. But recall that P^H and $\tilde P^H$ are H-stably conjugate. Thus $\varphi_{P^H,\tilde P^H}$ is defined and since $\varphi_{\tilde P^H,P'}=\varphi_{P^H,P'}\circ\varphi_{\tilde P^H,P^H}$ we have $\varphi_{P^H,\tilde P^H}(y)=\tilde y$. But Y contains only one element per stable class, which forces

$y = \tilde{y}$, and so $\varphi_{P^H,\tilde{P}^H}(y) = y$, i.e. P^H and $\tilde{P}^H$ are H^y-stably conjugate. This conclude the proof of the claim about the map p.

Consider a triple (y, ξ, P^H) contributing to (6.5) and let P' be its image under p. We focus on the part of (6.5) given by

$$\langle \mathrm{inv}(\gamma', \gamma), \widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1} \sum_{z \in H_y(F)_{\mathrm{sr}}/\mathrm{st}} \Delta_{0,y,\xi}(z, \gamma'_u) \frac{D_{H_y}(z)^2}{D_{G_{\gamma'_s}}(\gamma'_u)^2}$$
$$\sum_{Q^H} R(H_y, S_{Q^H}, 1)(z) \tag{6.6}$$

The map $\varphi_{\gamma',\gamma}$ defines an inner twist $G_{\gamma'_s} \to G^u_{\gamma_s}$ and maps γ'_u to γ_u. From this it follows that $D_{G_{\gamma'_s}}(\gamma'_u) = D_{G^u_{\gamma_s}}(\gamma_u)$, and $\mathrm{inv}(\gamma', \gamma) = \mathrm{inv}(\gamma'_u, \gamma_u) = \mathrm{inv}(X', X)$, where $X' = \log(\gamma'_u)$, $X = \log(\gamma_u)$. All z which are preimages of γ'_u are topologically unipotent, so we may restrict the sum over z to the topologically unipotent elements. Put $Z = \log(z)$. We will use Lemma 6.3.1 with $G' = G_{\gamma'_s}$ and $H' = H_y$. By [Kot86, Prop. 7.1] these groups are unramified and come with fixed hyperspecial maximal compact subgroups. We see that (6.6) equals

$$\langle \mathrm{inv}(X', X), \widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1} \sum_{Z \in \mathfrak{h}_y(F)_{\mathrm{sr}}/\mathrm{st}} \Delta_{0,y,\xi}(Z, X') \frac{D_{\mathfrak{h}_y}(Z)}{D_{\mathfrak{g}_{\gamma'_s}}(X')}$$
$$\sum_{Q^H} R(H_y, S_{Q^H}, 1)(z) \tag{6.7}$$

The function

$$\Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(Z, X) := \Delta_{0,y,\xi}(Z, X')\langle \mathrm{inv}(X', X), \widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}$$

is a transfer factor for $(\mathfrak{g}^u_{\gamma_s}, \mathfrak{h}_y, \varphi_{\gamma',\gamma} \circ \xi)$. Applying [DR09, Lem. 12.4.3] we conclude that (6.7) equals

$$\sum_{Z \in \mathfrak{h}_y(F)_{\mathrm{sr}}/\mathrm{st}} \Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(Z, X) \frac{D_{\mathfrak{h}_y}(Z)}{D_{\mathfrak{g}_{\gamma_s}}(X)} \sum_{Q^H} \epsilon(H_y, A_{H_y}) \widehat{\mu}^{H_y}_{Q^H}(Z) \tag{6.8}$$

Here $\widehat{\mu}^{H_y}_{Q^H}$ is the Fourier transform (with respect to the transfer to $\mathfrak{h}_y$ of the bilinear form B and the character ψ) of the orbital integral at Q^H on $\mathfrak{h}_y(F)$.

We will now apply [Wal97, Conj. 1.2], which is now a theorem due to the work of [Wal97], [Wal06], [HCL07] and [Ngo08]. According to it, (6.8) equals

$$\gamma_\psi(\mathfrak{g}^u_{\gamma_s})\gamma_\psi(\mathfrak{h}_y)^{-1}\epsilon(H_y, A_{H_y}) \sum_Q \Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(P^H, Q)\widehat{\mu}^{G_{\gamma_s}}_Q(X) \tag{6.9}$$

where Q runs over a set of representatives for the conjugacy classes of regular semi-simple elements in $\mathfrak{g}_{\gamma_s}(F)$.

For a moment we consider the signs in (6.9). The group H_y contains S^H_0, which is an elliptic maximal torus of H. Thus the inclusion $Z_H \to Z_{H_y}$ restricts to an isomorphism $A_H \to A_{H_y}$. The group $G_{\gamma'_s}$ contains S_0, which is an elliptic maximal torus of G, and again we get an isomorphism $A_G \to A_{G_{\gamma'_s}}$. The group $G^u_{\gamma_s}$ is an inner twist of $G_{\gamma'_s}$ and so we have an isomorphism $A_{G_{\gamma'_s}} \to A_{G^u_{\gamma_s}}$. Finally since H is elliptic for G, the natural inclusion $Z_G \to Z_H$ restricts to an

isomorphism $A_G \to A_H$. All in all this gives an isomorphism $A_{H_y} \to A_{G^u_{\gamma_s}}$. Using this and the transitivity of the sign $\epsilon(\cdot,\cdot)$ we conclude

$$\epsilon(H_y, A_{H_y}) = \epsilon(H_y, G_{\gamma'_s})\epsilon(G_{\gamma'_s}, G^u_{\gamma_s})\epsilon(G^u_{\gamma_s}, A_{G^u_{\gamma_s}})$$

From [DR09, §12.3] we know

$$\epsilon(G_{\gamma'_s}, G^u_{\gamma_s}) = \gamma_\psi(\mathfrak{g}_{\gamma'_s})\gamma_\psi(\mathfrak{g}^u_{\gamma_s})^{-1}$$

while from Proposition 4.0.3 we know

$$\epsilon(H_y, G_{\gamma'_s}) = \gamma_\psi(\mathfrak{h}_y)\gamma_\psi(\mathfrak{g}_{\gamma'_s})^{-1}$$

It follows that (6.9) equals

$$\sum_Q \Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(P^H, Q)\epsilon(G^u_{\gamma_s}, A_{G^u_{\gamma_s}})\widehat{\mu}_Q^{G_{\gamma_s}}(X) \tag{6.10}$$

where Q runs over the same set as in (6.9).

Now there is a natural injection from the set of $G^u_{\gamma_s}$-stable classes of regular semi-simple elements in $\mathfrak{g}^u_{\gamma_s}(F)$ stably conjugate to Q_0 to the set of $G_{\gamma'_s}$-stable classes of regular semi-simple elements in $\mathfrak{g}_{\gamma'_s}(F)$ stably conjugate to Q_0. If P' is not in the image of that injection, then (6.10) is zero. Otherwise let $P \in \mathfrak{g}_{\gamma_s}(F)$ be an element whose class maps to that of P'. Then (6.10) equals

$$\sum_Q \Delta_{0,y,\xi}(P^H, Q_0)\langle \mathrm{inv}(Q_0, Q), s_{q_0}\rangle^{-1}\epsilon(G^u_{\gamma_s}, A_{G^u_{\gamma_s}})\widehat{\mu}_Q^{G_{\gamma_s}}(X) \tag{6.11}$$

where Q runs over the set of $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

The torus $S_0 \subset G_{\gamma'_s}$ and the element Q_0 satisfy the requirements of Lemma 6.3.2. Moreover the element Q satisfies the requirements of [DR09, Lem 12.4.3] on the element X_S. Thus (6.11) equals

$$\sum_Q \langle \mathrm{inv}(Q_0, Q), s_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u) \tag{6.12}$$

where Q runs over the same set as in (6.11).

To recapitulate, for a triple (y, ξ, P^H) contributing to (6.5) there are two possibilities. Either $P' = p(y, \xi, P^H)$ does not lie in the image of the natural injection from the regular semi-simple stable classes in $\mathfrak{g}^u_{\gamma_s}(F)$ to those in $\mathfrak{g}_{\gamma'_s}(F)$ given by the inner twist $\varphi_{\gamma',\gamma} : G_{\gamma'_s} \to G^u_{\gamma_s}$, in which case the summand corresponding to that triple is zero. Or it does lie in that image, and if P is an element of the stable class in $\mathfrak{g}^u_{\gamma_s}(F)$ which maps to that of P', then the summand of (6.5) corresponding to (y, ξ, P^H) equals (6.12).

After restricting the sums in (6.5) to the subset of triples (y, ξ, P^H) whose image under p lies in the image of the natural injection of stable classes provided by $\varphi_{\gamma',\gamma}$, we obtain a map

$$(y, \xi, P^H) \mapsto P$$

which is a surjection on the set of $G^u_{\gamma_s}$-stable classes of elements of $\mathfrak{g}^u_{\gamma_s}(F)$ which are stably conjugate to Q_0, and the fiber of that surjection through (y, ξ, P^H) is a torsor under $\pi_0(H^y(F))$. This of course follows from the corresponding property of the map p.

Before we apply this to the expression (6.5), we need to note that if (y, ξ, P^H) maps to P, then since $\varphi_{P,P^H}(\gamma_s) = y$ we have

$$\begin{aligned}
[\varphi_{Q_0^H,P^H}]_*\theta_0^H(y) &= [\varphi_{P^H,P}]_*[\varphi_{Q_0^H,P^H}]_*\theta_0^H(\gamma_s) \\
&= [\varphi_{Q_0,P}]_*[\varphi_{Q_0^H,Q_0}]_*\theta_0^H(\gamma_s) \\
&= [\varphi_{Q_0,P}]_*\theta_0(\gamma_s)
\end{aligned}$$

where the last equality follows from Lemma 6.1.3.

With this in mind, we see that (6.5) equals

$$\begin{aligned}
\epsilon_L(V,\psi)\epsilon(H,A_H) \quad \cdot \quad & \sum_P [\varphi_{Q_0,P}]_*\theta_0(\gamma_s) \\
& \sum_Q \langle \mathrm{inv}(Q_0,Q), s_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u) \qquad (6.13)
\end{aligned}$$

where P runs over a set of representatives for the $G^u_{\gamma_s}$-stable classes of elements in $\mathfrak{g}^u_{\gamma_s}(F)$ which are G^u-stably conjugate to Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

Again using the transitivity of $\epsilon(\cdot,\cdot)$ and the isomorphism $A_G \cong A_H$ we can write

$$\epsilon(H,A_H) = \epsilon(H,G)\epsilon(G,A_G)$$

and thus using Proposition 4.0.3 we see that (6.13) equals

$$\epsilon(G,A_G)\sum_P [\varphi_{Q_0,P}]_*\theta_0(\gamma_s)\sum_Q \langle \mathrm{inv}(Q_0,Q), s_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u) \qquad (6.14)$$

with both sums as in (6.13). By Proposition 5.3.2 this is the left hand side of Equation (3.2). This completes the proof of Theorem 3.4.2. $\square$

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